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Involution words: Counting problems and connections to Schubert calculus for symmetric orbit closures

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A B S T R A C T

Involution words are variations of reduced words for involutions in Coxeter groups, first studied under the name of "admissible sequences" by Richardson and Springer. They are maximal chains in Richardson and Springer's weak order on involutions. This article is the first in a series of papers on involution words, and focuses on their enumerative properties. We define involution analogues of several objects associated to permutations, including Rothe diagrams, the essential set, Schubert polynomials, and Stanley symmetric functions. These definitions have geometric interpretations for certain intervals in the weak order on involutions. In particular, our definition of "involution Schubert polynomials" can be viewed as a Billey–Jockusch–Stanley type formula for cohomology class representatives of O_n- and Sp_{2n}-orbit closures in the flag variety, defined inductively in recent work of Wyser and Yong. As a special case of a more general theorem, we show that the involution Stanley symmetric function for the longest element of a finite symmetric group is a product of staircase-shaped Schur functions. This implies that the number of involution words for the longest element of a finite symmetric group is

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equal to the dimension of a certain irreducible representation of a Weyl group of type B.

Contents

1. Introduction

1.1. Involution words

Let (W, S) be a Coxeter system and define $\mathcal{I} = \mathcal{I}(W) = \{x \in W : x = x^{-1}\}$ to be the set of involutions in W. A *reduced word* for an element $w \in W$ is a sequence $(s_1, s_2, \dots, s_k)$ with $s_i \in S$ of shortest possible length such that $w = s_1 s_2 \cdots s_k$. An *involution word* for an element $z \in \mathcal{I}$ is a sequence $(s_1, s_2, \dots, s_k)$ with $s_i \in S$ of shortest possible length such that

$$z = (\cdots((1 \rtimes s_1) \rtimes s_2) \rtimes \cdots) \rtimes s_k \tag{1.1}$$

where for $g \in W$ and $s \in S$ we let $g \rtimes s$ be either gs (if s and g commute) or sgs (if $sg \neq gs$). When $g \in \mathcal{I}$, the element $g \rtimes s$ is also an involution. Less obviously, every $z \in \mathcal{I}$ has at least one involution word with the convention that the empty sequence $\emptyset$ is the unique involution word of the identity element $1 \in \mathcal{I}$. We write $\mathcal{R}(w)$ for the set