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# Planar arcs ☆

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ABSTRACT

Let $p$ denote the characteristic of $\mathbb{F}_q$, the finite field with $q$ elements. We prove that if $q$ is odd then an arc of size $q+2-t$ in the projective plane over $\mathbb{F}_q$, which is not contained in a conic, is contained in the intersection of two curves, which do not share a common component, and have degree at most $t+p^{\lfloor \log_p t \rfloor}$, provided a certain technical condition on $t$ is satisfied. This implies that if $q$ is odd then an arc of size at least $q-\sqrt{q}+\sqrt{q}/p+3$ is contained in a conic if $q$ is square and an arc of size at least $q-\sqrt{q}+\frac{7}{2}$ is contained in a conic if $q$ is prime. This is of particular interest in the case that $q$ is an odd square, since then there are examples of arcs, not contained in a conic, of size $q-\sqrt{q}+1$, and it has long been conjectured that if $q \neq 9$ is an odd square then any larger arc is contained in a conic.

These bounds improve on previously known bounds when $q$ is an odd square and for primes less than 1783. The previously known bounds, obtained by Segre [26], Hirschfeld and Korchmáros [17] [18], and Voloch [32] [33], rely on results on the number of points on algebraic curves over finite fields, in particular the Hasse–Weil theorem and the Stöhr–Voloch theorem, and are based on Segre's idea to associate an algebraic curve in the dual plane containing the tangents to an arc. In this paper we do not rely on such theorems, but use a new approach starting from a scaled coordinate-free version of Segre's lemma of tangents.

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Arcs in the projective plane over $\mathbb{F}_q$ of size $q$ and $q+1$, $q$ odd, were classified by Segre [25] in 1955. In this article, we complete the classification of arcs of size $q-1$ and $q-2$. The main theorem also verifies the MDS conjecture for a wider range of dimensions in the case that $q$ is an odd square. The MDS conjecture states that if $4 \leqslant k \leqslant q-2$, a $k$-dimensional linear MDS code has length at most $q+1$. Here, we verify the conjecture for $k \leqslant \sqrt{q} - \sqrt{q}/p + 2$, in the case that $q$ is an odd square.



## 1. Introduction

Let $\mathbb{F}_q$ denote the finite field with $q$ elements and let $p$ denote the characteristic of $\mathbb{F}_q$. Let $\mathrm{PG}(k-1,q)$ denote the $(k-1)$-dimensional projective space over $\mathbb{F}_q$.

An *arc* of $\mathrm{PG}(k-1,q)$ is a set of points, any $k$ of which span the whole space. An arc is *complete* if it cannot be extended to a larger arc. In this article we will be interested in arcs in $\mathrm{PG}(2,q)$, which we call *planar arcs*. A planar arc is defined equivalently as a set of points, no three of which are collinear. One can project any higher dimensional arc $A$ to a planar arc of size $|A|-k+3$, by choosing any $k-3$ points of the arc and quotienting by the subspace that they span. Therefore, the results contained in this article have implications for all low-dimensional arcs.

Arcs not only play an important role in finite geometry but also forge links to other branches of mathematics. The matrix whose columns are vector representatives of the points of an arc, generates a linear maximum distance separable code, see [21, Chapter 11] for more details on this. Other areas in which planar arcs play a role include the representation of matroids, see [23], Del Pezzo surfaces over finite fields, see [5], bent functions, see [22, Chapter 7] and pro-solvable groups, see [13].

In 1955, Beniamino Segre published the article [24], which contains his now celebrated theorem that if $q$ is odd then a planar arc of size $q+1$ is a conic. He went further in his 1967 paper [26] and considered planar arcs of size $q+2-t$ and proved that the set of tangents, when viewed as a set of points in the dual plane, is contained in a curve of small degree $d$. Specifically, if $q$ is even then $d=t$ and if $q$ is odd then $d=2t$.

His starting point, which will also be our starting point, was his lemma of tangents. Lemma 12 is a simplification of the coordinate-free version of Segre's original lemma of tangents which first appeared in [3].

Here, we do not apply Segre's lemma of tangents in the dual setting, nor combine it with interpolation, as was the approach in [3]. Our aim here is to prove Theorem 16, which maintains that there is a polynomial $F(X,Y)$, where $X=(X_1,X_2,X_3)$ and $Y=(Y_1,Y_2,Y_3)$, homogeneous of degree $t$ in both $X$ and $Y$, which upon evaluation in one of the variables at a point $a$ of the arc or at least a large subset of the arc, factorises into $t$ linear forms whose kernels are precisely the tangents to the arc at $a$.