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Maps related to polar spaces preserving a Weyl distance or an incidence condition

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A B S T R A C T

Let Ω_i and Ω_j be the sets of elements of respective types i and j of a polar space Δ of rank at least 3, viewed as a Tits-building. For any Weyl distance δ between Ω_i and Ω_j, we show that δ is characterised by i and j and two additional numerical parameters k and ℓ. We consider permutations ρ of $\Omega_i \cup \Omega_j$ that preserve a single Weyl distance δ. Up to a minor technical condition on ℓ, we prove that, up to trivial cases and two classes of true exceptions, ρ is induced by an automorphism of the Tits-building associated to Δ, which is always a type-preserving automorphism of Δ (and hence preserving *all* Weyl-distances), unless Δ is hyperbolic, in which case there are outer automorphisms. For each class of exceptions, we determine a Tits-building Δ' in which Δ naturally embeds and is such that ρ is induced by an automorphism of Δ'. At the same time, we prove similar results for permutations preserving a natural incidence condition. These yield combinatorial characterisations of all groups of algebraic origin which are the full automorphism group of some polar space as the automorphism group of many bipartite graphs.

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1. Introduction

Let Δ be a polar space of rank n with $n \geq 3$, with T its set of types and Ω_s its set of singular subspaces of type s (the type of a singular subspace is its dimension, except for the maximal singular subspaces of a hyperbolic quadric). The following situation is the central theme of some recent papers: Define some natural (adjacency) relation $\sim$ on Ω_s and determine the full automorphism group of the corresponding graph $(\Omega_s, \sim)$, hoping for the full automorphism group of Δ. For instance, Liu, Ma and Wang [21] essentially prove that when Δ is a finite unitary polar space, s is arbitrary but not maximal, and adjacency is "being incident with common singular subspaces of types $s-1$ and $s+1$", then the automorphism group of $(\Omega, \sim)$ coincides with the full automorphism group of the polar space. Zeng, Chai, Feng and Ma [29] prove the same thing for finite symplectic polar spaces. Pankov [23] shows this for general polar spaces, and points out the only exception, namely the polar space related to the triality quadric, where also trialities and dualities preserve this adjacency relation on the set of lines of the polar space (however, implicitly, this result was known long before, see Section 5). M. Pankov, K. Prazmovski and M. Zynel [24] show for an arbitrary polar space Δ and arbitrary s that, when adjacency is "being incident with a common singular subspace of type $s-1$", then the automorphism group of $(\Omega, \sim)$ coincides with the full automorphism group of the polar space (without exception). Huang and Havlicek [18] develop a technique that can be applied to this problem when the adjacency relation is given by "opposition" (see below for the precise definition of this notion). However, their result can not be applied to all polar spaces. Kasikova and Van Maldeghem [19] solve the case of opposition for all polar spaces and all possible types (pointing out several exceptions to the expectation of getting the full automorphism group of the polar space). Huang [16,17] shows that for many polar spaces, when s is maximal and adjacency is given by "intersecting in a singular subspace of type at most some fixed number", the automorphism groups of the graph and the polar space coincide. Liu, Pankov and Wang [22] treat the case where adjacency is given by "being incident with a common singular subspace of type $s-1$ and not with one of type $s+1$", and also the case where adjacency is defined as "being contained in a unique maximal singular subspace". In the present paper we consider adjacency relations that contain and generalise all previously mentioned relations. Moreover, we consider these relations between singular subspaces of possibly different types, which gives rise to bipartite graphs and yields slightly more general results and more counter examples. We note that the adjacency relations in [24,16,17] express an intersection property of the singular subspaces in question, while the adjacency relations in [21,29,18,19,22] express a certain *Weyl distance* in the associated Tits-building.

Hence we study permutations of $\Omega_i \cup \Omega_j$, for $i, j \in \mathsf{T}$ (note that in most cases i, j represent dimensions, only when Δ is hyperbolic there are two types of $(n-1)$-dimensional subspaces; hence, in general, $|i|$ and $|j|$ denote the corresponding dimensions), preserving either