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Lots of aperiodic sets of tiles

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ABSTRACT

Aperiodic tiling — a form of complex global geometric structure arising through locally checkable, constant-time matching rules — has long been closely tied to a wide range of physical, information-theoretic, and foundational applications, but its study and use has been hindered by a lack of easily generated examples. Through readily generalized, robust techniques for controlling hierarchical structure, we increase the catalogue of explicit constructions of aperiodic sets of tiles hundreds-fold, in lots, easily assembled and configured from atomic subsets of 211 tiles, enforcing 25,380 distinct "domino" substitution tiling systems.

0. Introduction

An **aperiodic set of tiles** is one that may be used to tile the plane, but only non-periodically — a form of complex global geometric structure arising through locally checkable, constant-time matching rules.

The very existence of aperiodic sets of tiles is implied by the undecidability of the "domino" (or "tiling") problem, that no algorithm can ever decide whether any given set of tiles can be used to form a tiling of the plane. Hao Wang opened this discussion in 1961 [17], in the context of his work on one of the then-remaining open cases of Hilbert's

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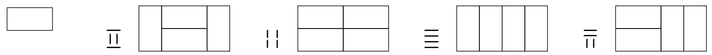

Fig. 1. Four configurations of domino tiles, and suggestive notation for referring to them.

Entscheidungsproblem ("Is a given first order logical formula satisfiable?"). Wang conjectured that the domino problem is decidable, citing the self-evident implausibility of any existence of aperiodic sets of tiles! Fortuitously, Wang was incorrect and Robert Berger soon showed the domino problem undecidable [1], producing the first aperiodic set as a tool in his proof. Since this initial construction, about forty more aperiodic sets of tiles have been explicitly described, most found by mysterious art. These aperiodic sets of tiles have long been closely tied to a wide range of physical, information-theoretic, dynamical and foundational applications, in a range of geometric and combinatorial settings — see [5] for background, supporting and bibliographic material. However the study and use of aperiodicity through local rules has been hindered by a lack of easily generated examples.

Through readily generalized, robust techniques for controlling hierarchical structure, we increase the catalogue of explicit constructions of aperiodic sets of tiles hundreds-fold, in lots, easily assembled and configured from smaller atomic subsets, industrializing their production and flexibly enforcing a range of hierarchical, substitution tiling needs at reasonable cost, an example of control one might routinely expect. Such constructions may serve as scaffold for further applications and stimulate further development of the theory of matching rule tiling spaces.

Enforcing domino substitution tilings We follow the long thread from [1] onwards, constructing aperiodic sets of tiles that only admit hierarchical, and hence non-periodic, tilings (Fig. 2), in our case on **domino** tiles, 2×1 rectangles

Our aperiodic sets "enforce" "substitution rules", as in Fig. 2; we define these terms precisely in Section 1. Even the simplest non-trivial substitution rules on the domino are unexpectedly rich: of the four configurations, in Fig. 1 (together with the symbols we'll use to name them) only three of them are enough to specify well-defined tiling substitution rules and hence tilings (see Fig. 3). Each of the first three rules can be iterated in only one way. (The dynamics of the $\underline{\mathrm{II}}$ (or "table") tiling substitution system, at left of Fig. 1, and of the L- (or "chair") substitution of Fig. 2 were studied in [13].)

However the fourth rule is not yet well-defined: The domino tile itself has more symmetry than the fourth configuration, and so there is an ambiguity when we try to iterate, and this is more so when we try a second time. In order to give a well-defined rule we must give the specific motions that we are allowed to use to place each child supertile into its parent, so that we know which end is which as we iterate the rule. We address this by framing each supertile, with the markings of Section 4.1.

By specifying which "pieces" — specified by "atomic symbols" — of substitution rules we will allow (Fig. 3), in other words, in which orientations we will allow children to be