Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Annals of Pure and Applied Logic 134 (2005) 122–168
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Linear realizability and full completeness for typed lambda-calculi

Samson Abramsky[a,1], Marina Lenisa[b,*]

[a] *Oxford University Computing Laboratory, Wolfson Building, Parks Road, OX1 3QD, United Kingdom*
[b] *Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze 206, 33100 Udine, Italy*

Received 1 December 2001; received in revised form 1 August 2004; accepted 1 August 2004
Available online 25 November 2004

Communicated by I. Moerdijk

Abstract

We present the model construction technique called *Linear Realizability*. It consists in building a category of *Partial Equivalence Relations* over a *Linear Combinatory Algebra*. We illustrate how it can be used to provide models, which are *fully complete* for various typed λ-calculi. In particular, we focus on special Linear Combinatory Algebras of *partial involutions*, and we present PER models over them which are fully complete, inter alia, w.r.t. the following languages and theories: the fragment of System F consisting of ML-types, the *maximal theory* on the simply typed λ-calculus with *finitely many* ground constants, and the maximal theory on an *infinitary* version of this latter calculus.

Keywords: Typed lambda-calculi; ML-polymorphic types; Linear logic; Hyperdoctrines; PER models; Geometry of Interaction; (Axiomatic) Full completeness

* Corresponding author. Tel.: +39 0432 558417; fax: +39 0432 558499.
E-mail addresses: Samson.Abramsky@comlab.ox.ac.uk (S. Abramsky), lenisa@dimi.uniud.it (M. Lenisa).
[1] Tel.: +44 (0)1865 283558; fax: +44 (0)1865 273839.

0168-0072/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2004.08.003

Introduction

The aim of this paper is to illustrate the technique of *Linear Realizability* for building *fully complete* models for various typed λ-calculi. This paper combines and expands previous works by the authors, [8–11].

A *categorical* model of a type theory (or logic) is said to be *fully complete* [6] if, for all *types* (*formulae*) A, B, all morphisms $f : [\![A]\!] \to [\![B]\!]$, from the interpretation of A into the interpretation of B, are denotations of a *proof-term* of the entailment $A \vdash B$, i.e. if the interpretation function from the category of syntactical objects to the category of denotations is *full*. The notion of full completeness is a counterpart to the notion of *full abstraction* for programming languages. Besides full completeness, one can ask the question of whether the theory induced by a model $\mathcal{M}$ coincides precisely with the syntactical theory or whether more equations are satisfied in $\mathcal{M}$. A model $\mathcal{M}$ is called *faithful* if it realizes exactly the syntactical theory.

A fully complete model indicates that there is a very tight connection between syntax and semantics. Equivalently, one can say that the term model has been made into a mathematically respectable structure.

Over the past decade, Game Semantics has been used successfully by various authors to define fully complete models for various fragments of Linear Logic, and to give fully abstract models for many programming languages, including PCF [7,24], and other functional and non-functional languages. In this paper, we propose the technique of *linear realizability* as a valid and less complex alternative to Game Semantics in providing fully complete and fully abstract models for typed λ-calculi.

The linear (linear affine) realizability technique amounts to constructing a category of *Partial Equivalence Relations (PERs)* over a *Linear Combinatory Algebra, LCA* (Affine Combinatory Algebra, ACA). This category turns out to be *linear* (affine), and to form an *adjoint model* with its co-Kleisli category. The notion of Linear (Affine) Combinatory Algebra introduced by the first author [2] refines the standard notion of Combinatory Algebra, in the same way in which intuitionistic linear (affine) logic refines intuitionistic logic. The construction of PER models from LCAs (ACAs) described in this paper is quite simple and clear, and it yields models with *extensionality* properties, thus avoiding extra quotienting operations, which are often needed in defining game categories and models. Many examples of linear combinatory algebras arise in the context of Abramsky's categorical version of Girard's *Geometry of Interaction*, [5,2,1,13].

In order to illustrate the technique of linear realizability, we present a number of case studies of fully complete models for various typed λ-calculi. These models arise from special ACAs of *partial involutions*, and they are defined in the co-Kleisli category of the category of PERs on them. The models which we study are the following:

(1) A fully complete (and necessarily faithful) model for the fragment of System F (with the $\beta\eta$-theory) consisting of *ML-polymorphic types*. ML-types are universal closures of simple types, i.e. types of the form $\forall X_1 \dots X_n.T$, where T is $\forall$-free.

(2) A fully complete model for the *maximal theory* $\approx$ on the simply typed λ-calculus with two ground constants $\perp, \top$. This model turns out to be fully complete also for an *infinitary* version of this calculus.