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Subgroups of the additive group of a separably closed field

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Abstract

We study the infinitely definable subgroups of the additive group in a separably closed field of finite positive imperfection degree. We give some constructions of families of such subgroups which confirm the diversity and the richness of this class of groups. We show in particular that there exists a locally modular minimal subgroup such that the division ring of its quasi-endomorphisms is not a fraction field of the ring of its definable endomorphisms, and that in contrast there exist $2^{\aleph_0}$ pairwise orthogonal locally modular minimal subgroups whose induced structure is exactly that of an $\mathbb{F}_p$-vector space. We also show that there exist infinitely many pairwise orthogonal subgroups of infinite U-rank. Furthermore, these constructions are carried out in the additive group considered as a module over its ring of definable endomorphisms.

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0. Introduction

We consider the class of infinitely definable subgroups (of a finite cartesian product) of the additive group in a separably closed field L of characteristic p and of finite positive

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imperfection degree ν (i.e. $[L : L^p] = p^\nu$). Note that every infinitely definable subgroup of a cartesian product of $(L, +)$ is definably isomorphic to a subgroup of $(L, +)$ because $(L^{\times p^{\nu k}}, +)$ is definably isomorphic to $(L, +)$ for every $k > 0$. Moreover all commutative infinitely definable subgroups of exponent p arise in this manner.

Our basic motivation for studying this class is that it is quite rich, hence a good place to look for examples and counter-examples for some still mysterious phenomena in the context of stability. Recall that, in contrast with the cases of algebraically closed fields (of any characteristic) and of differentially closed fields of characteristic 0, this class includes locally modular minimal additive subgroups. Moreover, in [3], Bouscaren and Delon answer positively the question of the existence of a non-thin minimal subgroup of $(L, +)$ (i.e. a minimal subgroup with infinite transcendence degree). Some of the main questions we had in mind when we began the study of this class were the following:

- What are the induced structures on these groups when they are one-based?
- What are the quasi-endomorphism division rings and the endomorphism subrings of these groups when they are minimal?
- Can we find $\aleph_0$-categorical groups?
- Can we find ranked groups of infinite U-rank?

In Section 2, we show that, as in algebraically closed fields, every infinitely definable subgroup (of a cartesian product) of the additive group is defined by additive equations.

In Section 3, we study the module structure of $(L, +)$ over the ring of all endomorphisms definable over an elementary substructure K. In particular, we observe that if G is a one-based infinitely definable subgroup (of a cartesian product) of $(L, +)$ over K, then the structure induced on G by the field L is exactly the structure induced by the module structure of $(L, +)$.

It is well known that the geometry associated with a locally modular minimal group G is the geometry of a vector space over the quasi-endomorphism ring of G. However the geometry does not give the full induced structure. In fact, the relation between the geometry and the induced structure turns out to be complicated in the case of a bounded exponent group (see [15]). In Section 4, we show that there exists a locally modular minimal subgroup of $(L, +)$ whose quasi-endomorphism division ring is not a field of fractions of its endomorphism subring. In the class of minimal subgroups of $(L, +)$, we can see the local modularity on the quasi-endomorphism division ring but not necessarily on the endomorphism subring. Finally, we construct a family of $\aleph_0$-categorical minimal subgroups of $(L, +)$ whose induced structure is exactly that of an $\mathbb{F}_p$-vector space.

In Section 5, we answer positively the question of the existence of ranked subgroups of infinite U-rank.

1. Preliminaries

We assume that the reader is familiar with the basic notions and facts of stability theory (see [4] or [21]). We will recall most of the definitions and results on stable groups which we use. Others, including basic facts about the generics in a stable group, can be found in [22,21] or [24].