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Annals of Pure and Applied Logic 131 (2005) 287–301

ANNALS OF PURE AND APPLIED LOGIC

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Completeness of S4 with respect to the real line: revisited☆

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Received 1 October 2001; received in revised form 30 November 2003; accepted 3 June 2004

Available online 2 September 2004

Communicated by S.N. Artemov

Abstract

We prove that **S4** is complete with respect to Boolean combinations of countable unions of convex subsets of the real line, thus strengthening a 1944 result of McKinsey and Tarski (Ann. of Math. (2) 45 (1944) 141). We also prove that the same result holds for the bimodal system **S4** + **S5** + **C**, which is a strengthening of a 1999 result of Shehtman (J. Appl. Non-Classical Logics 9 (1999) 369).

MSC (1991): 03B45; 06E25; 54C10

Keywords: Modal logic; Topology; Boolean algebras with operators; Completeness

1. Introduction

It was shown in McKinsey and Tarski [8] that every finite well-connected topological space is an open image of a metric separable dense-in-itself space. This together with the finite model property of **S4** implies that **S4** is complete with respect to any metric separable dense-in-itself space. Most importantly, it implies that **S4** is complete with respect to the real line $\mathbb{R}$. Shehtman [13] strengthened the McKinsey and Tarski result by showing that

☆ The research described in this publication was made possible in part by Award No. 3303 of the Georgian Research and Development Foundation (GRDF) and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).
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doi:10.1016/j.apal.2004.06.003

every finite connected space is an open image of a (connected) metric separable dense-in-itself space. (That every finite connected space is an open image of a Euclidean space was first established in Puckett [11].) As a result, Shehtman obtained that in the language enriched with the universal modality $\forall$ the complete logic of a connected metric separable dense-in-itself space is the logic **S4** + **S5** + **C**, where **S4** + **S5** is Bennett's logic [2] (being **S4** for $\Box$, **S5** for $\forall$, plus the bridge axiom $\forall\varphi \to \Box\varphi$) and **C** is the connectedness axiom $\forall(\Diamond\varphi \to \Box\varphi) \to (\forall\varphi \vee \forall\neg\varphi)$.

The original proof of McKinsey and Tarski was quite complicated. The later version in Rasiowa and Sikorski [12] was not much more accessible. Recently Mints [10] and Aiello et al. [1] obtained simpler model-theoretic proofs of completeness of **S4** with respect to the Cantor space $\mathcal{C}$ and the real line $\mathbb{R}$. In this paper we give yet another, more topological, proof of completeness of **S4** with respect to $\mathbb{R}$. It is not only more accessible than the original proof, but also strengthens both the McKinsey and Tarski, and Shehtman results.

The paper is organized as follows. In Section 2 we recall a one-to-one correspondence between Alexandroff spaces and quasi-ordered sets; we also recall the modal systems **S4**, **S4** + **S5** and **S4** + **S5** + **C**, and their algebraic semantics. In Section 3 we give a simplified proof that a finite well-connected topological space is an open image of $\mathbb{R}$. It follows that **S4** is complete with respect to Boolean combinations of countable unions of convex subsets of $\mathbb{R}$, which is a strengthening of the McKinsey and Tarski result. As a by-product, we obtain a new proof of completeness of the intuitionistic propositional logic **Int** with respect to open subsets of $\mathbb{R}$, and completeness of the Grzegorczyk logic **Grz** with respect to Boolean combinations of open subsets of $\mathbb{R}$. In Section 4 we give a simplified proof that a finite topological space is an open image of $\mathbb{R}$ iff it is connected. Consequently, we obtain that **S4** + **S5** + **C** is complete with respect to Boolean combinations of countable unions of convex subsets of $\mathbb{R}$, which is a strengthening of the Shehtman result. We conclude the paper by mentioning several open problems.

2. Preliminaries

2.1. Topology and order

Suppose X is a topological space. For $A \subseteq X$ we denote by $\overline{A}$ the closure of A, and by $Int(A)$ the interior of A. We recall that A is *dense* if $\overline{A} = X$, and that A is *nowhere dense* or *boundary* if $Int(A) = \emptyset$. The definition of closed and open subsets of X is usual. We call a subset of X *clopen* if it is simultaneously closed and open. The space X is called *connected* if $\emptyset$ and X are the only clopen subsets of X; it is called *well-connected* if there exists a least nonempty closed subset of X. It is obvious that every well-connected space is connected, but the converse is not necessarily true. We call X an *Alexandroff space* if the intersection of any family of open subsets of X is open. Obviously every finite space is an Alexandroff space. For two topological spaces X and Y, a continuous map $f : X \to Y$ is called *open* if the f-image of every open subset of X is an open subset of Y. Thus, f is an open map iff it *preserves* and *reflects* opens.

Suppose X is a nonempty set. A binary relation $\leq$ on X is called a *quasi-order* if $\leq$ is reflexive and transitive; if in addition $\leq$ is antisymmetric, then $\leq$ is called a *partial order*. If $\leq$ is a quasi-order on X, then X is called a *quasi-ordered set* or simply a *qoset*; if $\leq$ is