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MULTIPLICATION OF WEAK FUNCTIONS [1]

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Abstract This paper is a contionuation of [1] and is concerned with multiplication of weak functions. Here the weak functions are treated as generalized expansions in Hermite functions in [2].

Key words Multiplication, weak functions, generalized numbers

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1 Introduction

In [1], the ideas of generalized numbers and generalized weak functions were introduced and weak functions were treated as generalized expansions in Hermite polynomials. In the present paper, we will treat weak functions as generalized expansions in Hermite functions as in [2]. For completeness, we give the necessary ideas below.

Definition 1 Generalized numbers

We call the formal number series

$$\sum_{n=0}^{\infty} a_n$$

a generalized number, and denote it by

$$a = \sum_{n=0}^{\infty} a_n. \tag{1.1}$$

The m-th partial sum of (1.1) $\sum_{n=0}^{m} a_n$

is denoted by $(a)_m$, and we write

$$(a)_m \to a.$$

Definition 2 Weak Function

We call the generalized Hermite expansion

$$\sum_{n=0}^{\infty} a_n \psi_n(x) \tag{1.2}$$

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a weak function, where

$$\psi_n(x)=\frac{H_n(x)}{C_n}\mathrm{e}^{-\frac{|x|^2}{2}},\qquad H_n(x)=(-1)^n\mathrm{e}^{x^2}\frac{\mathrm{d}^n\mathrm{e}^{-x^2}}{\mathrm{d}x^n},\qquad C_n^2=2^n n!\sqrt{\pi}.$$

The series (1.2) is always weakly convergent over

$$K=\mathrm{Span}\{\psi_n(x),\ n=0,1,2,\cdots\}$$

with the scalar product $\langle f,\bar g\rangle=\int f(x)\bar g(x)\mathrm{d}x$.

We will give some weak functions below.

Example 1 The identity function 1.

$$\begin{aligned}1&=\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty}\psi_n(t)\mathrm{d}t\psi_n(x)=\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty}\psi_{2n}(t)\mathrm{d}t\psi_{2n}(x)\\&=\sum_{n=0}^{\infty}\frac{1}{C_{2n}}\int_{-\infty}^{+\infty}H_{2n}(t)\mathrm{e}^{-\frac{t^2}{2}}\mathrm{d}t\psi_{2n}(x)\\&=\sqrt{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{C_{2n}}H_{2n}(0)\psi_{2n}(x)\\&=\sqrt{2\pi}\sum_{n=0}^{\infty}\psi_{2n}(0)(-1)^n\psi_{2n}(x).\end{aligned}\tag{1.3}$$

Example 2 The function $\mathrm{sgn}x=\begin{cases}1, & x>0,\\ -1, & x<0.\end{cases}$

We have

$$\mathrm{sgn}\ x=\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty}\mathrm{sgn}\ t\psi_{2n+1}(t)\mathrm{d}t\psi_{2n+1}(x)=2\sum_{n=0}^{\infty}(-1)^n\alpha_{2n+1}\psi_{2n+1}(x),\tag{1.4}$$

where

$$\alpha_{2n+1}=\int_{-\infty}^{+\infty}\frac{\psi_{2n+1}(t)}{t}\mathrm{d}t.$$

Example 3 The function $\frac{1}{x}$.

We have

$$\frac{1}{x}=\sum_{n=0}^{\infty}\alpha_{2n+1}\psi_{2n+1}(x).\tag{1.5}$$

where

$$\alpha_{2n+1}=\int_{-\infty}^{+\infty}\frac{\psi_{2n+1}(x)}{x}\mathrm{d}x=\frac{(-1)^n2^{2n+1}n!}{C_{2n+1}}\int_{-\infty}^{+\infty}L_n^{\frac12}(x^2)\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x.$$

But

$$\begin{aligned}\int_{-\infty}^{+\infty}L_n^{\frac12}(x^2)\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x&=\int_0^{\infty}t^{-\frac12}L_n^{\frac12}(t)\mathrm{e}^{-\frac{t}{2}}\mathrm{d}t\\&=\frac{\Gamma(n+\frac32)\sqrt2}{n!\Gamma(\frac32)}F(-n,\frac12,\frac32,2)=\frac{\Gamma(n+\frac32)2\sqrt2}{n!}\int_0^1(1-2u^2)^n\mathrm{d}u\\&=\frac{\Gamma(n+\frac32)}{n!}J_n.\end{aligned}$$