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A comparison of two complexes

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ABSTRACT

We prove the conjecture of Lusztig in [5, Section 4]. Given a reductive group over $\overline{\mathbb{F}_q}[\varepsilon]/(\varepsilon^r)$ for some $r \geq 2$, there is a notion of a character sheaf defined in [4, Section 8]. On the other hand, there is also a geometric analogue of the character constructed by Gérardin [2]. The conjecture in [5, Section 4] states that the two constructions are equivalent, which Lusztig also proved for $r = 2, 3, 4$. Here we generalize his method to prove this conjecture for general r. As a corollary we prove that the characters derived from these two complexes are equal.

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1. Introduction

The purpose of this paper is to prove the conjecture in [5, Section 4]. We first recall its setting. Let $\mathbf{k}$ be an algebraic closure of a finite field $\mathbb{F}_q$ where q is a power of prime p and G be a connected reductive algebraic group over $\mathbf{k}$. Let T be a maximal torus and B a Borel subgroup of G which contains T. Also we let U be the unipotent radical of B. We denote the Lie algebra of G, B, T, U by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{n}$, respectively.

For a fixed integer $r \geq 2$, we define $G_r := G(\mathbf{k}[\varepsilon]/(\varepsilon^r))$ where ε is an indeterminate. Note that it has a natural algebraic group structure over $\mathbf{k}$ which is no longer reductive. Similarly, we define $B_r := B(\mathbf{k}[\varepsilon]/(\varepsilon^r)), T_r := T(\mathbf{k}[\varepsilon]/(\varepsilon^r))$, and $U_r := U(\mathbf{k}[\varepsilon]/(\varepsilon^r))$, which are again considered as algebraic groups over $\mathbf{k}$.

Throughout this paper, we assume $p \geq r$ and thus we have an isomorphism of varieties

$$G \times \mathfrak{g}^{r-1} \xrightarrow{\simeq} G_r : (x, X_1, \cdots, X_{r-1}) \mapsto x e^{\varepsilon X_1} \cdots e^{\varepsilon^{r-1} X_{r-1}}. \tag{1}$$

Note that $e^{\varepsilon X_1}, \cdots, e^{\varepsilon^{r-1} X_{r-1}}$ are well-defined since $p \geq r$. If G is abelian (thus so is $\mathfrak{g}$) then (1) is also an isomorphism of algebraic groups.

Now we define

$$\tilde{G}_r = \{(B_r g, g') \in (B_r \backslash G_r) \times G_r \mid g g' g^{-1} \in B_r\}$$

and consider the following diagram

$$T_r \xleftarrow{\tilde{\tau}} \tilde{G}_r \xrightarrow{\tilde{\pi}} G_r$$

where the morphisms on the diagram are defined as follows.

$$\tilde{\pi}(B_r g, g') = g', \qquad \tilde{\tau}(B_r g, g') = \sigma(g g' g^{-1})$$

Here $\sigma : B_r \to T_r$ is the composition of the quotient morphism $B_r \to B_r/U_r$ and the inverse of $T_r \hookrightarrow B_r \to B_r/U_r$.

For any variety X over $\mathbf{k}$, we denote by $\mathcal{D}(X)$ the bounded constructible derived category of ℓ-adic sheaves where $\ell \neq p$ is a fixed prime. One of the main goals in [5] is to show that for a generic character sheaf $\mathcal{L}$ on T_r, $\tilde{\pi}_! \tilde{\tau}^* \mathcal{L}$ is an intersection cohomology complex. (For the definition of character sheaves on T_r one may refer to [4].) One obstacle for this problem is that unlike the case $r = 1$, the morphism $\tilde{\pi} : \tilde{G}_r \to G_r$ is no longer proper.

In [5] one strategy is explained; first we show that there is a canonical isomorphism between this complex and a similar one given by a geometric analogue of the character in [2]. Then, we use Fourier–Deligne transform to show that the latter one is indeed an intersection cohomology complex up to shift. In this paper we are interested in comparing these two complexes, which is proven for $r = 2, 3, 4$ in [5, Section 4]. Here we generalize the method of [5] to arbitrary $r \geq 2$.