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MONOMIAL IDEALS WITH TINY SQUARES

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ABSTRACT. Let $I \subset K[x, y]$ be a monomial ideal. How small can $\mu(I^2)$ be in terms of $\mu(I)$, where μ denotes the least number of generators? It was widely expected that the inequality $\mu(I^2) > \mu(I)$ should hold whenever $\mu(I) \geq 2$. Here we disprove this expectation and provide a somewhat surprising answer to the above question.

1. Introduction

For an ideal I in a Noetherian ring R, let $\mu(I)$ denote as usual the least number of generators of I. If $\mu(I) = m$, how small can $\mu(I^2)$ be in terms of m? Obviously, in suitable rings with zero-divisors, we may have $\mu(I^2) = 0$. There even exist one-dimensional local domains $(R, \mathfrak{m})$ with the property that the square of their maximal ideal $\mathfrak{m}$ requires less generators than $\mathfrak{m}$ itself, see [1, 2]. However, if R is a regular local ring, or if R is a polynomial ring over a field K and I is a homogeneous ideal of R, it has been expected in [3] that the inequality $\mu(I^2) > \mu(I)$ should hold whenever $\mu(I) \geq 2$. This is indeed the case for any integrally closed ideal I in a 2-dimensional regular local ring. On the other hand, it is not too difficult to construct examples of monomial ideals I in a polynomial ring S with at least 4 variables such that $\mu(I^2) < \mu(I)$. However, these examples satisfy $\operatorname{height} I < \dim S$. So far no ideals I with $\mu(I^2) < \mu(I)$ were known for 2-dimensional regular rings. In this paper, we shall prove the following statements.

Theorem 1.1. *For every integer $m \geq 5$, there exists a monomial ideal $I \subset K[x, y]$ such that $\mu(I) = m$ and $\mu(I^2) = 9$.*

Moreover, this result is best possible for $m \geq 6$.

Theorem 1.2. *Let $I \subset K[x, y]$ be a monomial ideal. If $\mu(I) \geq 6$ then $\mu(I^2) \geq 9$.*

Here are some notation to be used throughout. We denote by $\mathcal{M}$ the set of monomials in $K[x, y]$, i.e.

$$\mathcal{M} = \{x^i y^j \mid i, j \in \mathbb{N}\}.$$

As usual, we view $\mathcal{M}$ as partially ordered by divisibility.

For a monomial ideal $J \subset K[x, y]$, we denote by $\mathcal{G}(J)$ its unique minimal system of monomial generators. It is well known that $\mathcal{G}(J)$ is of cardinality $\mu(J)$ and consists of all monomials in J which are minimal under divisibility, i.e.

$$\mathcal{G}(J) = \big(\mathcal{M} \cap J\big) \setminus \big(\mathcal{M} \cap J\big)\mathcal{M}^*$$

where $\mathcal{M}^* = \mathcal{M} \setminus \{1\}$.