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Fractal curves and rugs of prescribed conformal dimension

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ABSTRACT

We construct Jordan arcs of prescribed conformal dimension which are "minimal for conformal dimension," meaning the Hausdorff and conformal dimensions are equal. These curves are used to design fractal rugs, similar to Rickman's rug, that are also minimal for conformal dimension. These fractal rugs could potentially settle a standing conjecture regarding the existence of metric spaces of prescribed topological conformal dimension.

1. Introduction

Let (X, d) be a metric space. The subscripts of dim indicate the type of dimension, and we set $\dim \varnothing = -1$ for every dimension.

Quasisymmetric maps form an interesting intermediate class lying between homeomorphisms and bi-Lipschitz maps [6,8]. Topological dimension is invariant under homeomorphisms, and Hausdorff dimension is bi-Lipschitz invariant. Conformal dimension classifies metric spaces up to quasisymmetric equivalence [11]:

Definition 1.1. The *conformal dimension of X* is

$$\dim_C X = \inf\{\dim_H f(X) : f \text{ is quasisymmetric}\}.$$

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It is clear from the definition that conformal dimension is invariant under quasisymmetric maps, and hence under bi-Lipschitz maps.

Pansu introduced conformal dimension in 1989 [13], and the concept has been widely studied since. The primary applications of the theory of conformal dimension are in the study of Gromov hyperbolic spaces and their boundaries. The boundary of a Gromov hyperbolic space admits a family of metrics which are not bi-Lipschitz equivalent, but quasisymmetrically equivalent. Consequently, the conformal dimension of the boundary is well-defined, unlike its Hausdorff dimension [11]. Recent advancements involving applications of conformal dimension are exposed in [2] and [3]. Determining the conformal dimension of the Sierpinskí carpet (denoted $\dim_C SC$) is an open problem, but in [9] Keith and Laakso proved that $\dim_C SC < \dim_H SC$. Kovalev proved a conjecture of Tyson: conformal dimension does not take values strictly between 0 and 1 [10]. In [7] Hakobyan proved that if $E \subset \mathbb{R}$ is a uniformly perfect middle-interval Cantor set, then $\dim_H E = \dim_C E$ if and only if $\dim_H E = 1$.

Definition 1.2. A metric space X is called *minimal for conformal dimension* if $\dim_C X = \dim_H X$.

In [4] topological conformal dimension was defined; it is an adaptation of *topological Hausdorff dimension* which was defined in [1] as

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d - 1 \text{ for all } U \in \mathcal{U}\}.$$

Definition 1.3. The topological conformal dimension of X is

$$\dim_{tC} X = \inf\{d : X \text{ has basis } \mathcal{U} \text{ such that } \dim_C \partial U \leq d - 1 \text{ for all } U \in \mathcal{U}\}.$$

There is a key difference between conformal dimension and tC-dimension. Lower bounds for the former can be obtained through the presence of "diffuse" families of curves, while diffuse families of surfaces provide lower bounds for the latter. For precise statements, see Theorem 4.5 in [4] and Proposition 4.1.3 in [11]. While Fact 4.1 in [4] shows $\dim_{tC} X \in \{-1, 0, 1\} \cup [2, \infty]$, it is unknown whether tC-dimension attains all values in $[2, \infty]$.

The following conjecture was posed in [4]:

Conjecture 1.4. *For every $d \in [2, \infty]$ there is a metric space X with $\dim_{tC} X = d$.*

In this paper we provide examples of fractal spaces that could potentially settle Conjecture 1.4. To this end, it seems appropriate to consider topological squares that are not quasisymmetrically equivalent to $[0, 1]^2$. A classical fractal of this kind is *Rickman's rug*, which is the cartesian product of the von Koch snowflake with the standard unit interval. In general, a *fractal rug* is a product space of the form $R_d = V_d \times [0, 1]$, where V_d is a Jordan arc (a space homeomorphic to $[0, 1]$) with $d = \dim_C V_d$. At present, we do not have the tools necessary to determine the tC-dimensions of these fractals, but we suspect that $\dim_H R_d = \dim_{tC} R_d$. This would be consistent with the fact that R_d is minimal for conformal dimension, which follows from a result of Bishop and Tyson [11].

Suppose that one prescribes $d > 1$, and considers a Jordan arc V_d that enjoys the minimality property $\dim_H V_d = \dim_C V_d$. In this case, it would follow that $\dim_H R_d = d + 1$. If the conjectured equality $\dim_H R_d = \dim_{tC} R_d$ were to hold, we would then have $\dim_{tC} R_d = 1+d$, which would provide an affirmative answer to the question of existence in Conjecture 1.4.

In Section 3 we discuss fractal rugs and their dimensions in the context of Conjecture 1.4. In Section 4 we construct the Jordan arcs that are discussed in Section 3, which is the main result of the paper:

Theorem 1.5. *For every $c \geq 1$ there is a Jordan arc Λ with $\dim_C \Lambda = c$.*