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Ideals of nowhere dense sets in some topologies on positive integers

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ABSTRACT

We investigate the ideals of nowhere dense sets in three topologies on $\mathbb{N}$ (namely, the Furstenberg's, Golomb's, and Kirch's topology) related to arithmetic progressions. In particular, we explore relationships between these ideals, and show that each of them has a topological representation and cannot be extended to a summable ideal. Moreover, we study a related notion of Marczewski–Burstin countable representability.

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0$ – the set of non-negative integers. Following [14], for all $a, b \in \mathbb{N}$ the symbol $\langle b, a\rangle$ stands for the infinite arithmetic progression with the initial term b and the difference a:

$$\langle b, a\rangle = \{an + b : \ n \in \mathbb{N}_0\} = \{b, \ b + a, \ b + 2a, \ \ldots\}.$$

We use the symbol (a, b) to denote the greatest common divisor of a and b. The letter $\mathbb{P}$ symbolizes the set of all prime numbers and $\Theta(a)$ stands for the set of all prime factors of $a \in \mathbb{N}$. By $\mathbb{SF}$ let us denote the set of *square-free numbers* (i.e., numbers not divisible by any square greater than 1):

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$$\mathbb{SF} = \{1, 2, 3, 5, 6, 7, 10, 11, \ldots\}.$$

By *squareful numbers* we mean numbers which are not square-free, i.e., numbers for which an exponent of some prime factor is at least 2.

Let $[\mathbb{N}]^\omega$ denote the family of all infinite subsets of $\mathbb{N}$.

By treating the power set $\mathcal{P}(\mathbb{N})$ as the space $2^\mathbb{N}$ of all functions $f\colon \mathbb{N} \to 2$ (equipped with the product topology, where each space $2 = \{0, 1\}$ carries the discrete topology) and identifying subsets of $\mathbb{N}$ with their characteristic functions, we can talk about descriptive complexity of subsets of $\mathcal{P}(\mathbb{N})$.

For any partition $\{X_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ a set $S \subseteq \mathbb{N}$ is called a selector of the partition $\{X_n\}_{n\in\mathbb{N}}$ if and only if $\forall_{n\in\mathbb{N}}\ |X_n \cap S| = 1$.

For other basic notions concerning set theory and topology see, e.g., [13].

1.1. Three topologies

One can consider three topologies on $\mathbb{N}$:

- *Furstenberg's topology* $\mathcal{T}_F$
  with the base $\mathcal{B}_F = \{\langle b, a\rangle :\ b \le a\}$,
- *Golomb's topology* $\mathcal{T}_G$
  with the base $\mathcal{B}_G = \{\langle b, a\rangle :\ (a, b) = 1,\ b < a\}$,
- *Kirch's topology* $\mathcal{T}_K$
  with the base $\mathcal{B}_K = \{\langle b, a\rangle :\ (a, b) = 1,\ b < a,\ a \in \mathbb{SF}\}$.

The topology $\mathcal{T}_F$ was introduced in 1955 by H. Furstenberg in [11]. With its use he presented an elegant topological proof of the existence of infinitely many prime numbers. In 1959, S. Golomb in [12] presented a similar proof using the topology $\mathcal{T}_G$ defined in 1953 by M. Brown in [7]. In 1969, A. Kirch in [14] defined the topology $\mathcal{T}_K$, weaker than the topology of Golomb. All of these topologies have recently been studied by P. Szczuka, e.g., in [21], [22], [23].

Actually, the Furstenberg's topology was originally defined on $\mathbb{Z}$, with the base consisting of all doubly infinite arithmetic progressions (from $-\infty$ to $+\infty$). It turned $\mathbb{Z}$ into a metrizable, zero-dimensional, and totally disconnected space. In this paper, in order to make our considerations more unified, we trim this topology to $\mathbb{N}$. Note that the main properties are preserved: being a Hausdorff, regular, and totally disconnected space is hereditary. $(\mathbb{N}, \mathcal{T}_F)$ also remains second-countable and thus, from the Tychonoff–Urysohn metrization theorem, we get that the space is metrizable. The requirement that $b \le a$ guarantees that every basic set is closed, so the space is still zero-dimensional.

The topologies of Golomb and Kirch both are Hausdorff but not regular, and connected – however, $\mathcal{T}_G$ is not locally connected, as opposed to $\mathcal{T}_K$.

1.2. Three ideals

An *ideal* on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$, closed under taking finite unions and subsets of its elements. We assume that an ideal is proper ($\neq \mathcal{P}(\mathbb{N})$) and contains all finite sets. By Fin we denote the ideal of all finite subsets of $\mathbb{N}$.

Obviously, in any (non-trivial[1]) topology, the nowhere dense sets form an ideal. Let us then define three ideals on $\mathbb{N}$:

[1] The topology should have a base consisting only of infinite sets – otherwise, the ideal of nowhere dense sets would not contain all finite sets.