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Selective separability on spaces with an analytic topology ☆

Javier Camargo [a], Carlos Uzcátegui [a,b,*]

[a] *Escuela de Matemáticas, Facultad de Ciencias, Universidad Industrial de Santander, Ciudad Universitaria, Carrera 27 Calle 9, Bucaramanga, Santander, A.A. 678, Colombia*
[b] *Centro Interdisciplinario de Lógica y Álgebra, Facultad de Ciencias, Universidad de Los Andes, Mérida, Venezuela*

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We study two form of selective separability, SS and SS^+, on countable spaces with an analytic topology. We show several Ramsey type properties which imply SS. For analytic spaces X, SS^+ is equivalent to have that the collection of dense sets is a G_δ subset of 2^X, and also equivalent to the existence of a weak base which is an F_σ-subset of 2^X. We study several examples of analytic spaces.

1. Introduction

In this paper we study some combinatorial properties of countable topological spaces. We will focus on spaces with a definable topology ([17,18]), that is to say, the topology of the space, viewed as a subset of 2^X, has to be a definable set. Typically, the topology will be assumed to be an analytic subset of 2^X. The interplay between combinatorial properties of a space and the descriptive complexity of the topology itself has shown to be quite fruitful [6,15,17–19].

A topological space X is *selectively separable*, denoted SS, if for any sequence $(D_n)_n$ of dense subsets of X there is a finite $a_n \subseteq D_n$ for all $n \in \mathbb{N}$ such that $\bigcup_n a_n$ is dense in X. This notion was introduced by Scheepers [14] and has received a lot of attention ever since (see for instance [1–6,9]). Bella et al. [4]

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* Corresponding author.
E-mail addresses: jcamargo@saber.uis.edu.co (J. Camargo), cuzcatea@saber.uis.edu.co (C. Uzcátegui).

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showed that every separable space with countable fan tightness is SS. On the other hand, Barman and Dow [1] showed that every separable Fréchet space is also SS. In section 3 we present several combinatorial properties which implies SS.

Shibakov [15] showed a stronger result when the topology of the space is analytic. He showed that any Fréchet countable space with an analytic topology has a countable π-base (and thus it is SS). The existence of a countable π-base provides a characterization of a property quite similar to SS, it is a property related to a game naturally associated to a selection principle. Let G_1 be the two player game defined as follows. Player I plays dense subsets of X and Player II picks a point from the set played by I. So a run of the game consists of a sequence of pairs (D_n, x_n) where each D_n is a dense set played by I and $x_n \in D_n$ is the response of II. We say that II wins if $\{x_n : n \in \mathbb{N}\}$ is dense. Scheepers [14] showed that X has a countable π-base if, and only if, II has a winning strategy for G_1. For the property SS a similar game, denoted G_{fin}, is defined as before but now player II picks a finite subset of the dense set played by I. A space X has the property SS^+, if II has a winning strategy for G_{fin}. We show that, for X with an analytic topology, the game G_{fin} is determined. We also show that SS^+ is characterized by the existence of an F_σ weak π-base (that is, an F_σ subset $\mathcal{P}$ of 2^X such that every set in $\mathcal{P}$ has non empty interior and every non empty open set contains a set from $\mathcal{P}$). This turns out to be also equivalent to having that the collection of dense subsets of X is a G_δ subset of 2^X. We compare this notion of a weak base with that of a σ-compactlike family introduced in [2]. Our characterization of SS^+ allows to show very easily that the product of two SS^+ spaces with analytic topology is also SS^+. A result that holds in general as shown by Barman–Dow [2]. However our proof is different. We analyze a space constructed by Barman–Dow [2] which is SS and not SS^+ and show it has an analytic topology and has countable fan tightness. Finally, in the last section of the paper we present several examples of countable spaces.

2. Preliminaries

An *ideal* on a set X is a collection $\mathcal{I}$ of subsets of X satisfying: (i) $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$. (ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. (iii) $\emptyset \in \mathcal{I}$. We will always assume that an ideal contains all finite subsets of X. If $\mathcal{I}$ is an ideal on X, then $\mathcal{I}^+ = \{A \subseteq X : A \notin \mathcal{I}\}$. Fin denotes the ideal of finite subsets of the non negative integers $\mathbb{N}$. We denote by $A^{<\omega}$ the collection of finite sequences of elements of A. If s is a finite sequence on A and $i \in A$, $|s|$ denotes its length and $s\widehat{\ }i$ the sequence obtained concatenating s with i. For $s \in 2^{<\omega}$ and $\alpha \in 2^{\mathbb{N}}$, let $s \prec \alpha$ if $\alpha(i) = s(i)$ for all $i < |s|$ and $[s] = \{\alpha \in 2^{\mathbb{N}} : s \prec \alpha\}$. If $\alpha \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, we denote by $\alpha \restriction n$ the finite sequence $(\alpha(0), \cdots, \alpha(n-1))$ if $n > 0$ and for $\alpha \restriction 0$ is the empty sequence. The collection of all $[s]$ with $s \in 2^{<\omega}$ is a basis of clopen sets for $2^{\mathbb{N}}$. Let $a \in \mathsf{Fin}$ and $A \subseteq \mathbb{N}$, we denote by $a \sqsubseteq A$ if a is an initial segment of A, i.e., $a = A \cap \{0, \cdots, n\}$, where $n = \max a$. For $A \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, we denote by A/m the set $\{n \in A : m < n\}$ and by $A \restriction m$ the set $A \cap \{0, \cdots, m-1\}$.

Let X be a topological space and $x \in X$. All spaces are assumed to be regular and T_1. A space is *crowded* if does not have isolated points. A collection $\mathcal{B}$ of non empty open sets is *a π-base*, if every non empty open set contains an element of $\mathcal{B}$. For every point x, we use the following ideal

$$\mathcal{I}_x = \{A \subseteq X : x \notin \overline{A \setminus \{x\}}\}.$$

The ideal of nowhere dense subsets of X is denoted by $\mathsf{nwd}(X)$. Now we recall some combinatorial properties of ideals. We put $A \subseteq^* B$ if $A \setminus B$ is finite.

($\mathbf{p}^+$) $\mathcal{I}$ is $\mathbf{p}^+$, if for every decreasing sequence $(A_n)_n$ of sets in $\mathcal{I}^+$, there is $A \in \mathcal{I}^+$ such that $A \subseteq^* A_n$ for all $n \in \mathbb{N}$. Following [11], we say that $\mathcal{I}$ is $\mathbf{p}^-$, if for every decreasing sequence $(A_n)_n$ of sets in $\mathcal{I}^+$ such that $A_n \setminus A_{n+1} \in \mathcal{I}$, there is $B \in \mathcal{I}^+$ such that $B \subseteq^* A_n$ for all n.