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Short communication

Analytical solutions for the flow depth of steady laminar, Bingham plastic tailings down wide channels

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ABSTRACT

At mid to high concentrations, fine mine tailings are non-Newtonian, and their rheology is commonly expressed as Bingham plastic. Discharges of such fine materials in tailing storage facilities form shallow channels. In this short communication, exact analytic expressions to relate the volume flow per unit width and the flow depth are derived for a Bingham plastic in terms of a newly-defined dimensionless parameter. Simplified approximations, valid for the near-plug and quasi-Newtonian fluid limits are proposed.

1. Introduction

Among the new trends in global mining is the need to reduce water consumption in arid areas given its increasing scarcity or cost, when transported from remote locations. This reality concurs with progressive decrease in ore grades (Ihle and Kracht, 2018), and consequently the need to process finer ores for optimal liberation. Low water content tailings —either thickened or paste—, can behave as non-segregating during transport (Simms, 2017). While they are discharged, they can form channels if they are thickened (Blight, 2010) or flow forming sheet-like structures if they are paste tailings (Robinsky, 1999). In this context, their high concentration nature makes them non-Newtonian, and it is common to assume that they flow as Bingham plastics (Huang and García, 1997; Sofrá and Boger, 2002; Blight, 2010). Both in the case of the self-organized flow after discharge in thickened tailings storage facilities or when highly viscous paste slurries are disposed as cones, the corresponding flow is often laminar throughout their whole trajectory or evolves from turbulent to laminar (Henriquez and Simms, 2009). As such discharges are unconfined, flow prediction requires setting a relationship between the mean flow velocity and the flow depth, given the particular rheological behavior of the slurry, in this case Bingham plastic. To this purpose, a number of authors have proposed models for various fluid types and channel geometries. In particular, Haldenwang (2003) and Alderman and Haldenwang (2007) present complete reviews of empirical and semi-empirical models, Javadi et al. (2015) analyze a new Reynolds number with previous experimental data, while a more recent account for developments in friction factor models in closed conduits is given in Carravetta et al. (2015). Once the model is set, either flow-depth or rheology is obtained by solving non-linear equations (except on the Newtonian case) or an inverse problem on the rheological parameters if they are unknown.

In the present short communication, explicit analytical and semi-analytical solutions for the relation between volume flow per unit width and the laminar discharge flow of a wide Bingham plastic channel is given, thus complementing a previous work for pipe flows (Ihle and Tamburrino, 2012), and serving as an alternative approach for Bingham rheological parameter determination.

2. Problem description

We consider the problem of a steady, uniform flow of a Bingham plastic down an inclined plane, assumed as an infinitely wide channel. Transient flow features such as the flow of the front of the discharge (commonly described using lubrication theory in Newtonian or non-Newtonian fluids as in, *e.g.*, Benjamin, 1957; Yih, 1963; Lister, 1992; Balmforth et al., 2006) or the presence of roll waves (Tamburrino and Ihle, 2013) are not considered herein. The shear rate-shear stress relationship of a Bingham plastic is given by (Liu and Mei, 1989):

$$\eta\frac{\partial U}{\partial z}=\begin{cases}0 & \text{if } |\tau|<\tau_0\\ \tau-\tau_0\,\mathrm{sgn}\left(\frac{\partial U}{\partial z}\right) & \text{if } |\tau|\geqslant\tau_0,\end{cases} \tag{1}$$

where U, η, τ and τ_0 are the main component of velocity, Bingham viscosity, shear stress and yield stress, respectively. The function sgn is

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defined as $\text{sgn}(x) = x/|x|$ if $x \neq 0$ and 0 otherwise. This model has been used to predict transient gold tailing discharges, where film flow assumptions can give an accurate description of the front advance of thin sheets on slopes (Liu and Mei, 1989; Henriquez and Simms, 2009). Given this type of fluid has a yield stress, this kind of flow can have sections where the shear stress is below τ_0 (Liu and Mei, 1989). The resulting, so-called plug flow, affects the mean velocity profile and therefore the total flow depth given a volume flow per unit width q_0, affecting the dynamical balance in the system. The critical depth h_p of a completely plugged layer corresponds to (e.g Griffiths, 2000):

$$h_p = \frac{\tau_0}{\rho_m g \sin\theta}, \tag{2}$$

where g is the magnitude of the gravity acceleration vector, ρ_m the density of the solid-liquid mixture and θ the slope of the plane. This condition is often used to estimate required tailing deposit capacities in tailing storage facilities (Robinsky, 1999), and is independent of viscosity given there is no shear within this layer.

3. Governing equations

Under the set of hypotheses denoted above, the flow is assumed slender and two-dimensional. The corresponding momentum equation on an infinite plane of slope θ is given by:

$$\rho_m\left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U}\right) = -\nabla p + \frac{\partial \tau}{\partial z}\hat{\imath} + g(\sin\theta\hat{\imath} + \cos\theta\hat{\boldsymbol{k}}), \tag{3}$$

where $\mathbf{U} = U\hat{\imath} + W\hat{\boldsymbol{k}}$ is the velocity vector, p the pressure and $(\hat{\imath}, \hat{\boldsymbol{k}})$ are unit vectors aligned with the (x, z) axes shown in Fig. 1. The corresponding boundary conditions are $\tau(z=0) = \rho_m g H\sin\theta$ and $\tau(z=H) = 0$ (Huang and García, 1997). The continuity of the shear stress and the existence of a yield stress implies that at some point there exists a critical height z^* where the shear stress is lower than τ_0. This point is denoted as $z^* = (1-\lambda)H$, with $\lambda < 1$. It is noted that when $\lambda \ll 1$, the effect of the yield stress is negligible, and the fluid is quasi-Newtonian. On the other hand, the case $\lambda \lessapprox 1$ corresponds to the situation when most of the flow is moving as a solid. If $\lambda = 1$, $H = h_p$ and the column remains stagnant. Under the latter assumptions, the velocity field $\mathbf{U} = U(z)$, corresponding to the solution of (3), reads:

$$u(z) = \begin{cases} \frac{U_p z}{(1-\lambda)H}\left[2 - \frac{z}{(1-\lambda)H}\right] & \text{if } 0 \leqslant z \leqslant (1-\lambda)H \\ U_p & \text{otherwise,} \end{cases} \tag{4}$$

with

$$U_p = \frac{\rho_m g(1-\lambda)^2 H^2 \sin\theta}{2\eta}. \tag{5}$$

Integrating the velocity profile, the corresponding average velocity is given by (e.g. Bird et al., 1987):

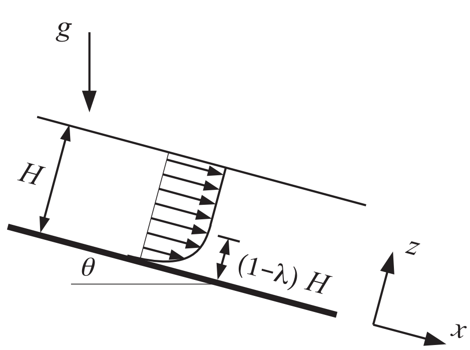

Fig. 1. Schematic of the problem. Plug flow exists for $(1-\lambda)H \leqslant z^* \leqslant H$.

$$\overline{U} = \frac{\tau_w H}{3\eta}\left[1 - \frac{3}{2}\frac{\tau_0}{\tau_w} + \frac{1}{2}\left(\frac{\tau_0}{\tau_w}\right)^3\right], \tag{6}$$

where $\tau_w = \rho_m g H \sin\theta$, is the shear stress at the bottom of the layer. This relation is similar to that found in other works for rectangular channels and laminar flow (see Alderman and Haldenwang, 2007).

The parameter λ could be also written as the ratio between the depth given by (2) and the flow depth:

$$\lambda = \frac{h_p}{H} = \frac{\tau_0}{H\rho_m g\sin\theta}. \tag{7}$$

From the right side of (7), also $\lambda = \tau_0/\tau_w$. Assuming the inflow q_0 is known, an integral volume conservation statement can be imposed to obtain an equation for the flow depth H, $q_0 = \int_0^H U(z')\mathrm{d}z'$. Using (7) in the latter expression yields the nondimensional equation

$$\mathscr{N} - (h-1)^2(2h+1) = 0, \tag{8}$$

where $h = H\rho_m g\sin\theta/\tau_0$ and

$$\mathscr{N} = \frac{6\eta q_0(\rho_m g\sin\theta)^2}{\tau_0^3} \tag{9}$$

is a dimensionless control parameter, which is related to the non-Newtonian characteristic of the fluid, where $\mathscr{N} \ll 1$ implies a strong influence of the yield stress, and $\mathscr{N} \gg 1$ a quasi-Newtonian fluid behavior. In particular, defining the Bingham number as $B = \frac{\tau_0}{\eta\overline{U}/H}$, interpreted as a dimensionless yield stress (Balmforth et al., 2006), it is straightforward to obtain that:

$$\mathscr{N} = \frac{6h^2}{B}. \tag{10}$$

Given flow and fluid properties, $\mathscr{N}$ is always positive. If $\mathscr{N} \leq 1$, (8) has three real roots. The solution for the present problem pose no ambiguity, as one root is negative and one of the positive solutions is lower than one, which is not compatible with the requirement that flow depth must exceed h_p implying, in dimensionless terms, that $h \geqslant 1$. Thus, for $0 \leqslant \mathscr{N} \leq 1$, the only meaningful solution is the largest root of (8):

$$h(0 \leqslant \mathscr{N} < 1) = \frac{1}{2} + \cos\left[\frac{1}{3}\arccos(2\mathscr{N} - 1)\right] \tag{11}$$

$$\approx 1 + \frac{\sqrt{3}}{3}\mathscr{N}^{1/2} - \frac{\mathscr{N}}{9} + \frac{5\sqrt{3}}{162}\mathscr{N}^{3/2} - \frac{8}{243}\mathscr{N}^2 + \mathscr{O}(\mathscr{N}^{5/2}) \tag{12}$$

The argument of the second term on the right hand side of (11) cannot be reduced further using trigonometric identities (Abramowitz and Stegun, 1965). The expression (12) shows the first terms of the corresponding Taylor expansion around $\mathscr{N} = 0$. This is plotted in Fig. 2, where it is shown that for $\mathscr{N} \ll 1$, $h - 1 \sim \mathscr{N}^{1/2}$. In dimensional terms, this means that

$$H_{\mathscr{N}\ll 1} \sim h_p + \left(\frac{2\eta q_0}{\tau_0}\right)^{1/2}. \tag{13}$$

Here, by virtue of (10) $\mathscr{N} \ll 1$ implies $B \gg 1$ or, in other words, a very high dimensionless yield stress. The last result is expressed in terms of B as $h \approx \left(1 - \sqrt{\frac{6}{B}}\right)^{-1}$. The second term on the right hand side of (13) naturally suggests itself as a flow length scale provided $[2\eta q_0(\rho_m g\sin\theta)^2]^{1/3} \ll \tau_0$. It is independent of the slope, indicating that for this, near-plugging regime, the sheared part of the velocity field strongly obeys a purely yield-viscous stress balance.

If $\mathscr{N} \geqslant 1$, there is one real root of (8) whose algebraic expression is given by

$$h(\mathscr{N} \geqslant 1) = \frac{1}{2^{2/3}}\left(h_+ + h_- + \frac{1}{2^{1/3}}\right) \tag{14}$$

with