Computer Aided Geometric Design 66 (2018) 19–30

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

The Fuss formulas in the Poncelet porism ☆

Waldemar Cieślak [a], Witold Mozgawa [b,*]

[a] *Lublin University of Technology, Department of Applied Mathematics, 20-618 Lublin, Nadbystrzycka 40, Poland*
[b] *Institute of Mathematics, Maria Curie-Skłodowska University, pl. M. Curie-Skłodowskiej 1, 20-031 Lublin, Poland*

ARTICLE INFO

Article history:
Received 20 December 2017
Received in revised form 30 July 2018
Accepted 31 July 2018
Available online 20 August 2018

Dedicated to Professor Jan Kisyński

Keywords:
Poncelet's porism

ABSTRACT

In this paper we give a proof of Poncelet's closure theorem for ring domains using elementary functions and a certain differential equation which has a solution with suitable geometric properties. We give a necessary and sufficient condition of existence of a constant solution of the equation which explains the phenomenon of the Poncelet porism. In the last section we present a method of determination of the Fuss formulas for an arbitrary natural n. Additionally this method allows us to find the Fuss formulas for closed n-gons with self-intersections.

1. Introduction

The story started with Jean-Victor Poncelet (July 1, 1788 – December 22, 1867) a French engineer, a mathematician, and a reviver of the projective geometry. As a military engineer, he served in Napoleon's campaign against the Russian Empire in 1812. Poncelet found and proved the theorem called *closure theorem* in 1813–1814 during his stay in Saratov after the Napoleon war.

After his return to France he published in 1822 a very informative book *Traité sur les propriétés projectives des figures* (Poncelet, 1865–1866) where he gave the proof of the closure theorem. His proof involved an argument which he called *principle of continuity* which was already considered doubtful by some contemporary mathematicians, but it led him to correct results. The theorem concerns conics and interscribed polygons.

Let us consider two ellipses C and D and let the ellipse C lie inside the second ellipse D. From any point M_1 on D, draw a tangent line to C and extend it to D in the opposite direction. From this point we draw another tangent line, etc. Thus we construct Poncelet's transverse $(M_1, l_1, M_2, l_2, M_3, l_3, \ldots)$ such that $M_i \in D$ and l_i is tangent to the ellipse C with $M_{i+1} = l_i \cap l_{i+1}$. We say that Poncelet's transverse closes after n steps if $M_{n+1} = M_1$.

With the above notations we have the following Poncelet's closure theorem.

Theorem 1.1 *(PCT). If a transverse starting at a point $M_1 \in D$ of the ellipse D closes after n steps then any Poncelet's transverse starting at any point of the ellipse D closes after n steps.*

Sometimes this theorem is called the Poncelet porism.

We recall PCT in the version considered in this paper. Let us fix a ring domain formed by two circles one within another. A closed polygon will be called Poncelet's polygon if it is described on the inner circle and circumscribed in the outer circle

☆ This paper has been recommended for acceptance by Johannes Wallner.

* Corresponding author.
E-mail addresses: izacieslak@wp.pl (W. Cieślak), mozgawa@poczta.umcs.lublin.pl (W. Mozgawa).

https://doi.org/10.1016/j.cagd.2018.07.006
0167-8396/

of a fixed ring domain. Poncelet proved that if by one point of the greater circle of a ring domain passes Poncelet's polygon then by each point of this circle always passes Poncelet's polygon. The proof given by Poncelet is a synthetic one, see Poncelet (1865–1866). In 1828 Jacobi gave an analytic proof of PCT and he used there elliptic functions, see Jacobi (1823). Schoenberg modified the Jacobi proof in 1983, see Schoenberg (1983).

The paper Bos et al. (1987) is a broad study on the Poncelet porism and it contains a contemporary proof of PCT. A contemporary contribution to Poncelet's porism is given in the articles of Griffiths and Harris in the papers Griffiths and Harris (1977, 1978) where the authors gave a modern proof of the closure theorem and exhibited the connections of PCT with elliptic curves. In Tabachnikov (1993) the author gave another proof using hyperbolic geometry.

Let us consider two circles C_R and C_r, where the circle C_r of radius r lie inside the second circle C_R of radius R in such a way that the distance between their centers is equal to a. We recall that

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

is a complete elliptic integral of the first kind and

$$\mathrm{sc}(x,k)=\tan\varphi,\ \text{where } x=\int_0^{\varphi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

is a Jacobi elliptic function, see Weisstein. The Jacobi condition for a n-transverse to be closed is

$$\mathrm{sc}\left(\frac{K(k))}{n},k\right)=\frac{c\sqrt{b^2-d^2}+b\sqrt{c^2-d^2}}{d(b+c)},$$

where $d=\frac{1}{R+a}$, $b=\frac{1}{R-a}$, $c=\frac{1}{r}$, $k^2=1-\exp\left(-2\cosh^{-1}\left(1+\frac{2c^2(d^2-b^2)}{d^2(b^2-c^2)}\right)\right)$.

Many mathematicians gave formulas without resorting to the elliptic functions e.g. Fuss (1797), Jacobi (1823), Steiner (1827), Dörrie (1965), Chaundy (1923), Richelot (1830), Radić (2010). Note that Kerawala (1947) gave several formulas but some of his formulas were incorrect. However, there is no compact algebraic formula relating n, R, r and a. We should emphasize that all known formulas involve algebraic functions, except for possibly their containing radicals.

For example,

1° for a triangle we have the Euler triangle formula $F_3(a,R,r)=0$, where

$$F_3(a,R,r)=a^2+2Rr-R^2, \tag{1.1}$$

Euler (1765), Steiner (1827), Hart (1858), Gabriel-Marie (1912), Kerawala (1947), Altshiller-Court (1952), Wells (1992),

2° for a quadrilateral we have the Fuss formula $F_4(a,R,r)=0$, where

$$F_4(a,R,r)=(R^2-a^2)^2-2r^2(R^2+a^2), \tag{1.2}$$

Fuss (1797, 1802), Davis, Durége (1878), Casey (1888), Gabriel-Marie (1912), Johnson (1929), Dörrie (1965),

3° for a pentagon we have $F_5(a,R,r)=0$, where

$$F_5(a,R,r)=a^6-2a^4rR+8a^2r^3R-3a^4R^2-4a^2r^2R^2+4a^2rR^3+3a^2R^4+4r^2R^4-2rR^5-R^6, \tag{1.3}$$

Steiner (1827), Richelot (1830) etc.

All the formulas above are related to convex polygons. The authors in Cieślak et al. (2013) have found the formula for a self-intersecting polygon. This formula was obtained in a way completely different from the one presented in this paper.

We note that there is an isomorphism between the Poncelet porism and Gelfand's question on the distribution of the leftmost digits of the numbers $2^n, 3^n, \ldots, 9^n$, see King (1994).

Fuss was the first mathematician who gave formulas without elliptic functions. For this reason, in this paper, we will use the formulation "the Fuss formula" for formulas without elliptic functions. It was shown in Cieślak and Szczygielska (2010) that there exists a relation between R, r, a and n where the elementary functions were used only. This result will be given in the next section. The aim of this paper is to give the Fuss formulas for an arbitrary natural number n for the convex polygons as well as for those with the self-intersections.