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Exact solutions for functionally graded micro-cylinders in first gradient elasticity

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ABSTRACT

An exact solution is obtained for a functionally graded (FG) micro-cylinder subjected to internal and external pressures. And in the current model, its material properties are assumed to be isotropic and exponentially-varying elastic modulus in radial direction and a material length scale parameter is incorporated to capture the size effect. To this end, a theoretical formulation including the effect of size dependency is derived in the framework of the first gradient elasticity and through Hamilton's principle. Then, a fourth-order governing ordinary differential equation (ODE) with variable coefficients is developed and the corresponding solution is rather difficult to be determined for inhomogeneous problem. Next, the efficient tensor algorithm operator is utilized to reduce the fourth-order homogeneous ODE to a second-order non-homogeneous one. Finally, by using method of variation of constant, the exact solution for FG micro-cylinder problem containing the material length scale parameter and power index is constructed perfectly, which is qualitatively different from existing Lamé's solution in classical elasticity. When ignoring the inhomogeneity of material, the newly obtained exact solution reduces to the ordinary one. The numerical results reveal that increasing characteristic length parameter leads to the decrease of the maximum radial and tangential stresses, and the power index has also a considerable effect on the stress distribution of FG micro-cylinders. A key physical insight that emerges from our analysis is that the newly obtained solution form can be applied directly to practical engineering structures.

1. Introduction

Recently, a novel composite material, termed as functionally gradient materials (FGMs) [1–3], has attracted many researchers because of their interesting characteristics. Due to its distinctive material properties varying its microstructure from one material to another continuously and smoothly with a specific gradient resulting in corresponding changes in the effective material properties (including elasticity modulus, shear modulus and material density) of the material through certain dimensions [4–9]. FGMs possess a number of advantages, including a potential reduction of in-plane and transverse stresses, reduced stress intensity factors, improved residual stress distribution and enhanced thermal properties [3,10–14]. Thus, the designed FG materials can be fabricated an optimum distribution of component materials for specific function and applications, i.e., space planes, space structures, turbine rotors, flywheels, gears, hollow metal nanowires and carbon nanotubes, which provide promising physical insight in composites [2,8,11,12]. In these inhomogeneous materials, the elastic and plastic properties of inhomogeneous cylinders (especially their mechanical behaviors of FG composites subjected to internal and external pressures), and the distribution of the matrix and their interactions with the matrix determine the overall physical properties of the composites. How to accurate understanding of the mechanical behaviors of these pressurized microstructures is a key problem. However, conducting experiments with microsize specimens under pressure is expensive and difficult, hence, the development of theoretical model with size dependency for the accurate analysis of their mechanical behaviors is of great importance for the design of advanced FG microcylinder-based composite materials and devices.

For the properties of homogenous cylinders, the problems have been solved completely. But finding solutions to elastic isotropic inhomogeneous FG thick-walled cylinder problems are not easy [4–7,15–20]. A few very restricted classes of inhomogeneous problems are solved in two general ways, one is the stress function and the other is the displacement function for the choice of the unknown function in the ordinary differential governing equations of FG cylinders. You et al. [17] transformed the governing equation in terms of the stress into Kummer's second order ordinary differential equation, an analytical elastic solution for pressurized hollow cylinders with internal FG coatings converged very quickly and had excellent accuracy. Li and Peng [18] chose the stress function in the ordinary differential equation of the axisymmetric FG cylinder, and reduced the FG hollow cylinder problem to solve a

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Fredholm integral equation, and approximately obtained the solution of stresses and displacements as series of Legendre polynomials. Tutuncn and Ozturk [19] obtained closed-form solutions for stresses and displacements in pressurized FG cylindrical and spherical vessels using the infinitesimal theory of elasticity by solving the governing equation of radial displacement, and shown the arbitrarily chosen power constants to demonstrate the effect of inhomogeneity on the stress distribution. Chen and Lin [20] adopted the displacement function as unknown in the governing equation, and a numerical solution of thick-walled cylinder and sphere problems was obtained and the boundary value problem was properly solved as well by using the transmission matrix.

But with the size reduction of micro- and nanostructures, conventional continuum elasticity the classical elasticity theory may be not applicable in understanding of the mechanical behaviors of the microstructural effect of the material [21]. To consider the intrinsic length scales that represent the measures of microstructure in their constitutive relations, a variety of new strain gradient theories have been developed to investigate such problems, i.e., couple stress elasticity theory [22–25], nonlocal strain gradient elasticity theory [26–28], and strain gradient elasticity theory [29]. Mehralian et al. [30] captured the size dependent buckling of anisotropic piezoelectric cylindrical shells by applying a modified couple stress theory, in which three length scale parameters were introduced. Faghidian exhibited the application value of Reissner variational principle in the framework of the nonlocal strain gradient theory of elasticity including two size-dependent characteristic parameters. Collin et al. [31] proposed the analytical solutions to the thick-walled cylinder problem for different loading conditions and Eshel and Rosenfeld [32] considered the stress-concentration problem of a circular cylindrical hole in a mechanically homogeneous, isotropic and centrosymmetric infinite elastic solid subjected to a field of uniaxial tension based on the general strain gradient elasticity theory, which contains seven material constants (two classical Lamé constants and five parameters associated with gradient terms). However, it should be noteworthy that the resulted governing equations are terribly complicated and analytical solutions for these equations are difficult to obtain even for a simple case, and meanwhile, the microstructure-dependent length scale parameters of material are difficult to determine. Hence, there is an impatient need to provide a complete, and rigorous strain gradient elasticity model to simplify the burdensome mechanical problem. By simplifying the special form of the general strain gradient elasticity theory [29], the first gradient elasticity [33] has evidently enjoyed great success so far, which reduces the elastic constants to three, with two being Lamé constants and the third one being the intrinsic length scale parameter introducing the microstructural effect. Gao and Park [34] reformulated variational formulation of the minimum total potential energy with considering the strain gradient elasticity energy based on a modified strain gradient elasticity theory (the first gradient elasticity), the displacement form of fourth-order ordinary differential equilibrium equation was obtained. A newly obtained displacement form of solution contains a material length scale parameter was presented to solve analytically the problem of a pressurized thick-walled cylinder. Zheng et al. [35] presented an analytical solution considering the microstructural and surface effects to investigate the axisymmetric deformation of circular nanotubes, which can be reduced to the solutions derived by Gao and Park [34]. However, the above pressurized cylinders are made of isotropic homogeneous materials which can only be optimized for their applications by material selection and very limited dimensional design. Sadeghi et al. [6] analyzed the FG micro-cylinders subjected to internal and external pressures based on the first gradient elasticity, and the problem solving procedure of the resulted fourth-order ordinary differential equilibrium equation is terribly intricated and unintelligible. Hosseini et al. [36] investigated strain gradient effect on the thermoelastic analysis of a rotating FG micro-cylinder. They solved the fourth-order ordinary differential equation in terms of mechanical displacement using a sophisticated and generalized differential quadrature method. However, the solutions are not exact and closely associated with number of discretization points, which requires less CPU time and computational effort. Conceptually, the problems in solid mechanics involving inhomogeneous media are relatively straightforward. Such problems can be formulated in terms of partial differential equations with variable coefficients by using the basic conservation laws. There has always been difficulty in developing general methods for solving specific boundary value problems. Therefore, simple and exact analytical solutions should be sought for the FG micro-cylinder problem so that can be conveniently used for the practical micro/nano-structure systems, which incorporate both the effect of inhomogeneity and microstructure-dependent.

For this purpose, a novel analytical approach is presented in this work to solve the elastic problem of pressurized inhomogeneous FG micro-cylinder. Upon employing the symmetry of the cylinder and the efficient tensor algorithm, the fourth-order homogeneous ordinary differential governing equation in terms of the displacement reduces to a second-order non-homogeneous one. Exact form solutions for FG micro-cylinders in first gradient elasticity will be obtained, and the distribution of the radial and tangential stresses can be determined. And numerical results are presented graphically to show the effect of gradient on the radial and tangential stresses and the microstructure-dependent effect has been successfully captured. The tangential stress exhibits a completely different response from that of a homogeneous micro-cylinder. In particular, the maximum tangential stress may occur at an interior point or at the outer surface.

2. First gradient elasticity formulation

In the non-classical continuum theory, Mindlin [29] first proposed the general form of strain gradient elasticity theory for capturing the size effect in microstructures. By simplifying the theory of Mindlin, Altan and Aifantis [33] redefined the variation of the elastic strain energy density function, which was a function of strain and second-order deformation gradient. Thus, the simplified elastic strain energy density function for isotropic and inhomogeneous materials can be written as [6,33–34]

$$w = w(\varepsilon_{ij}, \varepsilon_{ij,k}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + l^2\left[\frac{1}{2}\lambda\varepsilon_{ii,k}\varepsilon_{jj,k} + \mu\varepsilon_{ij,k}\varepsilon_{ij,k}\right], \tag{1}$$

where w is the elastic strain energy density function, λ and μ are the Lamé parameters as the functions of Cartesian coordinates, and l is the characteristic length parameter. ε_{ij} and $\varepsilon_{ij,k}$ are the components of the classical strain tensor and second-order deformation gradient tensor, respectively. And for small strain deformation, the strain is related to the displacement through

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

in which u_i is the total displacement filed caused by the elastic strain and $u_{i,j}$ is the derivative of u_i with respect to x_j.

Under the infinitesimal deformation assumption, the constitutive relations for microstructure-dependent, linearly elastic material can be readily written, in terms of the strain energy density, as [6,33]

$$\begin{aligned} \tau_{ij} &= \lambda\varepsilon_{ll}\delta_{ij} + 2\mu\varepsilon_{ij} = \tau_{ji}, \\ \mu_{ijk} &= l^2(\lambda\varepsilon_{ll,k}\delta_{ij} + 2\mu\varepsilon_{ij,k}) = \mu_{jik}, \\ \kappa_{ijk} &= \varepsilon_{ij,k} = \tfrac{1}{2}(u_{i,jk} + u_{j,ik}), \end{aligned} \tag{3}$$

where τ_{ij} and μ_{ijk} are the Cauchy stress tensor and double stress tensor, respectively. κ_{ijk} is the strain gradient tensor, δ_{ij} is the Kronecker delta. And here, the total stress can be defined [34]

$$\sigma_{ij} = \tau_{ij} - \mu_{ijk,k} = (\lambda\varepsilon_{ll}\delta_{ij} + 2\mu\varepsilon_{ij}) - l^2(\lambda\varepsilon_{ll,k}\delta_{ij} + 2\mu\varepsilon_{ij,k})_{,k}. \tag{4}$$

Furthermore, in the deformed configuration Ω, Cauchy's first law of motion yields the equilibrium equation by applying the principle of variation [31,34,37]

$$\sigma_{ij,j} + f_i = 0, \tag{5}$$