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Symmetry, mutual dependence, and the weighted Shapley values ☆

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Abstract

We pinpoint the position of the (symmetric) Shapley value within the class of positively weighted Shapley value to their treatment of symmetric versus mutually dependent players. While symmetric players are equally productive, mutually dependent players are only jointly (hence, equally) productive. In particular, we provide a characterization of the whole class of positively weighted Shapley values that uses two standard properties, efficiency and the null player out property, and a new property called superweak differential marginality. Superweak differential marginality is a relaxation of weak differential marginality (Casajus and Yokote, 2017). It requires two players' payoff for two games to change in the same direction whenever only their joint productivity changes, i.e., their individual productivities stay the same. In contrast, weak differential marginality already requires this when their individual productivities change by the same amount. The Shapley value is the unique positively weighted Shapley value that satisfies weak differential marginality.

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Keywords: TU game; Weighted Shapley values; Symmetry; Mutual dependence; Weak differential marginality; Superweak differential marginality

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1. Introduction

The (symmetric) Shapley value (Shapley, 1953b) probably is the most eminent one-point solution concept for cooperative games with transferable utility (TU games or simply games). Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985).

In order to account for asymmetries among players beyond the game itself, Shapley (1953a) already suggests weighted versions of his symmetric value, where these asymmetries are modelled by strictly positive weights for the players, the positively weighted Shapley values.[1] There exists a number of axiomatic foundations for the positively weighted Shapley values. Two types of axiomatizations can be distinguished: axiomatizations with exogenous weights and axiomatizations with endogenous weights. In axiomatizations with exogenous weights, the players' weights are explicitly given and the properties used may involve the weights (see, e.g., Hart and Mas-Colell, 1989; Calvo and Santos, 2000). In contrast, axiomatizations with exogenous weights aim at characterizing the whole class of positively weighted Shapley values, i.e., the properties used do not involve the weights, which implicitly are given by the solutions themselves (see, e.g., Kalai and Samet, 1987; Hart and Mas-Colell, 1989; Chun, 1991; Nowak and Radzik, 1995).

Recently, Casajus and Yokote (2017) suggest a characterization of the Shapley value by three properties of solutions: efficiency, the null player property, and weak differential marginality. Efficiency: The players' payoffs sum up to the worth generated by the grand coalition. Null player property: unproductive players, i.e., a null player obtain a zero payoff. Weak differential marginality: whenever two players' marginal contributions to coalitions not containing either of them change by the amount, then their payoffs change in the same direction, i.e., the signs of the changes have to coincide. This characterization, however, does not work for games with exactly two players. In order to obtain a characterization for all finite games, it suffices to replace the null player property by the null player out property: Whenever an unproductive player, i.e., a null player leaves a game, the remaining players' payoffs do not change.

In general, the positively weighted Shapley values fail weak differential marginality. Instead, they obey a relaxation of differential marginality called *superweak differential marginality*: whenever two players' marginal contributions to coalitions not containing either of them do not change, then their payoffs should change in the same direction. Note that while keeping the implication of weak differential marginality, superweak differential marginality strengthens its hypothesis. Instead of just requiring equal changes of the marginal contributions, it demands these changes to be zero.

As our main result, we show that the class of positively weighted Shapley values for variable finite player sets from an infinite universe of players is characterized by efficiency, the null player out property, and superweak differential marginality (Theorem 6). This way, the position of the Shapley value within the class of positively weighted Shapley values can be pinpointed to their treatment of symmetric players versus mutually dependent players. While two players are symmetric in a game when they are equally productive, i.e., their marginal contributions to coalitions not containing either of them coincide, they are mutually dependent when these marginal contributions are zero (Nowak and Radzik, 1995). Now, the hypothesis of weak differential marginality

[1] Later on, Kalai and Samet (1987) extend the class of positively weighted Shapley values by considering weight systems that allow for zero weights of the players.