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Generalized Haar wavelet operational matrix method for solving hyperbolic heat conduction in thin surface layers

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ABSTRACT

It is remarkably known that one of the difficulties encountered in a numerical method for hyperbolic heat conduction equation is the numerical oscillation within the vicinity of jump discontinuities at the wave front. In this paper, a new method is proposed for solving non-Fourier heat conduction problem. It is a combination of finite difference and pseudospectral methods in which the time discretization is performed prior to spatial discretization. In this sense, a partial differential equation is reduced to an ordinary differential equation and solved implicitly with Haar wavelet basis. For the pseudospectral method, Haar wavelet expansion has been using considering its advantage of the absence of the Gibbs phenomenon at the jump continuities. We also derived generalized Haar operational matrix that extend usual domain $(0, 1]$ to $(0, X]$. The proposed method has been applied to one physical problem, namely thin surface layers. It is found that the proposed numerical results could suppress and eliminate the numerical oscillation in the vicinity jump and in good agreement with the analytic solution. In addition, our method is stable, convergent and easily coded. Numerical results demonstrate good performance of the method in term of accuracy and competitiveness compare to other numerical methods.

Introduction

Heat conduction problem often arises in a variety of problems in various branches of science and engineering. When the heat flux or the temperature involved in the heat transfer process is not very high, or the phenomena occurring at a time scale smaller than the thermal relaxation time of the material are not of interest, these problems are best described and analyzed with the heat conduction equation based on Fourier's law, in which is in the parabolic-type heat conduction equation. This equation implies a presumption of infinite thermal propagation speed. Therefore, its prediction may underestimate the peak temperature during a rapid transient heat process. Classical diffusion theories may break down when one is interested in the transient problems in an extremely short period of time, in very high flux, or for very low temperature [1]. The simplest approach to construct a non-Fourier heat equation is through the modified equation by Vernotte [2] and Catteneo [3] to the classical heat diffusion equation based on Fourier's law, which includes a component recognising the finite speed of heat signals to be named as relaxation time often represented by τ. For example, earlier successful application of the hyperbolic heat equation used to predict the transient heat conduction process in chemical and process engineering [4], in the process of laser pulse heating [5], and in thin surface layers [6–8].

Numerous literature available for solving hyperbolic heat conduction. The major difficulty encountered in the numerical solution of the hyperbolic conduction is numerical oscillations in the vicinity of sharp discontinuities. Tamma and Railkar [9] was successfully overcome this problem by introducing specially tailored transfinite-element formulations for the hyperbolic heat conduction equation. Carey and Tsai [10] applied central and backward difference schemes to examine numerical oscillation errors at the reflected boundary. Chen and Lin [11] employed a new powerful hybrid technique based on the Laplace transform and control volume methods to solve the hyperbolic heat conduction equation and their numerical methods provides excellent results. They applied to various examples of physical problems to verify the accuracy of their method. Chen [6] solved the hyperbolic heat conduction equation in the thin surface layer using a hybrid method combined Laplace transform, weighing function scheme [12] and hyperbolic shape function.

Wensheng and Leigh [13] present anti-diffusive solutions to the hyperbolic heat transfer equation using Bokanowski et al. second-order method [14] and Xu and Shu's fifth-order method [15] in one dimensional and extend to two dimensional. Sarra [16] employed a numerical with less grid points compared to the finite-difference and the finite element method using the Chebyshev collocation method with the Gegenbauer reconstruction procedure. Glass et al. [17] used

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MacCormack's predictor-corrector scheme and finite-difference to reduce numerical oscillation at the front wave. The analytical solution of one dimension for hyperbolic heat conduction equation was given by Kao [8], Baumeister and Hamill [18], Ozisik and Vick [19] and Taitel [20]. Yen and Wu [21] studied a model of hyperbolic heat conduction in a finite slab with surface radiation and heat flux, involving nonlinearity in the boundary condition. The recent research of hyperbolic heat conduction can be found in Refs. [22–26] and it is still an active area of research.

Over the last 20 years, wavelet transforms have been applied extensively for applications in various fields like pattern recognition [27], data compression and signal processing [28]. However, the application of wavelet transform to the solution of the hyperbolic partial differential equation has been limited [29]. Wavelets with their highly localized functions in spatial dimension, which are of varying scales, have the potential to combine the advantages of both spectral and finite difference bases. Another good feature of using wavelets is that there is a class of fast algorithm based on the fast wavelet transform which may be used to speed up numerical schemes [30]. Historically, the wavelet transform was developed as an extension of Fourier transform in order to decompose the frequency content of a function in both spatial and frequency domain.

Focus in this research is on the pseudospectral method for spatial discretization, by Haar wavelet expansion to solve implicitly hyperbolic heat conduction in thin surface layers. Haar wavelet is the simplest wavelet and the only orthogonal wavelet that does not exhibit numerical oscillation near the jump discontinuity at any points [31,32]. This is the utmost concern when attempting to solve the hyperbolic-type heat conduction equation numerically. It will be a great advantage for our method to suppress oscillation that might appear, because the backward finite difference is being used to discretize time. To the best of our knowledge, the proposed method in this paper is the first attempt to solve the hyperbolic heat conduction equation. The analytical solution for prescribed wall temperature conditions have been given by Kao [8].

The first work in solving system analysis via Haar wavelet was led by Chen and Hsiao [33], who first derived a Haar operational matrix for the integrals of the Haar function vector and brought the application of Haar analysis into the dynamic systems. It can be found that Lepik [34–36] has established the Haar wavelet method to solve ordinary and partial differential equation and recently, solved PDE with two dimensional Haar wavelets [37]. Generalized Haar operational matrix is an extension work of Wu et al. [38] that covers the whole domain for its operational matrix. We derived the Haar operational matrix based on Wu et al. works but extending it using the generalized block pulse function operational matrix for integration done by Adem Kilicmen [39]. It is expedient to do this way as it will fit the expansion of Haar series in the interval $0 \leqslant x < X$.

Overview of Haar wavelet

In the following, we give a brief introduction to Haar wavelet function, its series expansion, matrix form and operational matrix. Many literatures have defined Haar wavelet operational matrix on the interval $[0, 1)$. Here we extend the usual defined interval to $[0, X)$ as the actual problem does not necessarily hold until 1 only. Finally, we introduced a new algorithm in order to solve the boundary value problem.

Haar Wavelet function

An analytic function $f(x)$ can be expanded in a series as

$$f(x) = \sum_{n}^{\infty} a_n \psi_n(x), \tag{1}$$

where $\varphi_n(x)$ is the basis in the Hilbert space $L^2(R)$ and a_n is the coefficient of the series. In this work an orthogonal function, namely Haar wavelet function is considered. The set of this function is a group of square wave in an interval of $(0, X]$ defined as

$$h_0(x) = \frac{1}{\sqrt{m}}, \quad (0 \leqslant x < X), \tag{2}$$

$$h_1(x) = \frac{1}{\sqrt{m}} \begin{cases} 1, & 0 \leqslant x < X, \\ -1, & \frac{X}{2} \leqslant x < X, \\ 0, & \text{elsewhere}, \end{cases} \tag{3}$$

and

$$h_i(x) = \frac{1}{\sqrt{m}} \begin{cases} 2^{\frac{j}{2}}, & \frac{k-1}{2^j}X \leqslant x < \frac{k-\frac{1}{2}}{2^j}X, \\ -2^{\frac{j}{2}}, & \frac{k-\frac{1}{2}}{2^j}X \leqslant x < \frac{k}{2^j}X, \\ 0, & \text{elsewhere}, \end{cases} \tag{4}$$

where $i = 1, 2, \ldots, m-1$, $m = 2^j (j = 0, 1, 2\ldots J)$ represents the level of wavelet and the resolution J is a positive integer. While j and k denoted the integer decomposition of the index i, for example $i = m + k-1$ in which $k = 1, 2, 3, \ldots, 2^j$. $h_0(x)$ is called scaling function, while $h_1(x)$ is called mother wavelet function. All the others following Haar wavelet functions are generated from mother wavelet function $h_1(x)$ with translation and dilation process i.e. the transform make the function as follows

$$h_i(x) = 2^{\frac{j}{2}} h_1(2^j x - k). \tag{5}$$

Haar wavelet functions are also orthogonal functions, so that it holds the property as below

$$\int_0^X h_p(x) h_q(x) dt = \begin{cases} \frac{X}{m}, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases} \tag{6}$$

The orthogonal set of the first four Haar function ($m = 4$) in the interval of $(0 \leqslant x < 1)$ can be shown in Fig. 1.

Haar series expansion

Any function $f(x) \in L^2([0, X))$ can be decomposed into Haar series and can be written as

$$f(x) = \sum_{i=0}^{\infty} c_i h_i(x). \tag{7}$$

If the function $f(x)$ may be approximated as a piecewise constant then the sum in Eq. (7) may be truncated after m terms in the form

$$f_m(x) \approx \sum_{i=0}^{m-1} c_i h_i(x), \tag{8}$$

where the Haar coefficient c_i are determined by

$$c_i = \frac{m}{X} \int_0^X f_m(x) h_i(x) dx. \tag{9}$$

If Eqs. (7) and (8) are the exact and approximate representation, respectively, then the corresponding error is defined as

$$e_m(x) = f(x) - f_m(x). \tag{10}$$

According to Habibollah et al. [40], they have shown that the square of the error norm for Haar wavelet approximation is first-order accurate as below

$$\|e_m(x)\| = O\left(\frac{1}{m}\right). \tag{11}$$

From Eq. (11), it is clear that the error is inversely proportional to the level resolution of Haar wavelet function. This implies that the Haar