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Generalized Log-sine integrals and Bell polynomials

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ABSTRACT

In this paper, we investigate the integral of $x^n \log^p(\sin(x))$ for natural numbers n and p. In doing so, we recover some well-known results and remark on some relations to the log-sine integral $\mathrm{Ls}_{n+p+1}^{(n)}(\theta)$. Later, we use properties of Bell polynomials to find an expression for the derivative of the central binomial and shifted central binomial coefficients as finite sums of polygamma functions and harmonic numbers.

1. Introduction and preliminaries

The Log-sine integral

$$\mathrm{Ls}_n(\theta) := -\int_0^\theta \left(\log\left|2\sin\left(\frac{x}{2}\right)\right|\right)^{n-1} dx$$

and its generalized expression

$$\mathrm{Ls}_n^{(m)}(\theta) := -\int_0^\theta x^m \left(\log\left|2\sin\left(\frac{x}{2}\right)\right|\right)^{n-m-1} dx$$

have been widely studied in previous papers (see [1–7]). The Log-sine integral has wide importance in many problems in mathematics and physics, including the calculations of higher order terms in the ϵ-expansion of Feynman diagrams. A very nice identity was given in [1] by expressing

$$S(k) := \sum_{n=1}^{\infty} \frac{1}{n^k \binom{2n}{n}}$$

as

$$S(k) = \frac{(-2)^{k-1}}{(k-2)!}\mathrm{Ls}_k^{(1)}\left(\frac{\pi}{3}\right), k \in \mathbb{N}.$$

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Evaluating Log-sine integrals can be difficult; this paper will shed light on a more elementary and perhaps quicker way of computing these integrals. We will focus on a similar integral,

$$F(n,m,z) := \int_0^z x^n \sin^{2m}(x)\, dx.$$

We can also define

$$G(n,m,z) := \int_0^z x^n \left(2\sin\left(\frac{x}{2}\right)\right)^{2m} dx,$$

and one can easily see that

$$G(n,m,2z) = 4^m 2^{n+1} F(n,m,z) \tag{1}$$

and

$$-\left.\frac{\partial^p G(n,m,z)}{\partial m^p}\right|_{(n,0,z)} = 2^p \mathrm{Ls}_{p+n+1}^{(n)}(z). \tag{2}$$

As we discuss results about $F(n,m,z)$, we will add in remarks for $G(n,m,z)$ and thus for the Log-sine integral. The goal of this paper is to find a formula for $I(n,p,z) := \int_0^z x^n \log^p\big(\sin(x)\big)\, dx$ for natural numbers n and p at certain z-values. This is the more general version of $\int_0^z x^n \log\big(\sin(x)\big)\, dx$, which has a known expression in terms of Clausen functions (see [8]). This could also be thought of as a variation of the generalized Log-sine integral that is related to $G(n,m,z)$ and $F(n,m,z)$. In this paper, we will use a different method from most; this method will be explained after we define several special functions. First, the Riemann zeta function and the polylogarithm function are defined by

$$\zeta(s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}, & \mathrm{Re}(s) > 1, \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, & \mathrm{Re}(s) > 0,\ s \neq 1, \end{cases} \tag{3}$$

and

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad n \in \mathbb{N} \setminus \{1\}, \quad |z| \le 1, \tag{4}$$

respectively. Euler discovered the now famous closed formula for $\zeta(2k)$, given by

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}, \quad k \in \mathbb{N}_0, \tag{5}$$

where B_n are the Bernoulli numbers, defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad |z| < 2\pi.$$

It is also clear that

$$\mathrm{Li}_n(1) = \zeta(n), \qquad \mathrm{Li}_n(-1) = -(1-2^{1-n})\zeta(n), \quad n \in \mathbb{N} \setminus \{1\}. \tag{6}$$

Next we introduce the generalized hypergeometric function

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+k-1) \tag{7}$$

is the Pochhammer symbol or rising factorial. If $a_1 = a_2 = \cdots = a_i = a$, we will use the notation ${}_pF_q(\{a\}^i, a_{i+1}, \ldots, a_p; b_1, b_2, \ldots, b_q; z)$. A special case used in the paper is

$${}_{q+1}F_q(\{1\}^{q+1}; \{2\}^q; z) = \sum_{k=0}^{\infty} \frac{(1)_k (1)_k \cdots (1)_k}{(2)_k (2)_k \cdots (2)_k} \frac{(1)_k z^k}{k!},$$