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Non-isotonic routing metrics solvable to optimality via shortest path

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ABSTRACT

It is a well-established fact that routing metrics cannot be optimized via shortest path algorithms unless they are both monotone and isotonic. In particular, monotonicity guarantees the convergence of the shortest path procedure, while isotonicity guarantees convergence to the optimal path. This paper presents a class of routing metrics that are *not* isotonic, yet can be solved to *exact* optimality via shortest path (or iterative shortest path) procedures. In particular, we consider routing metrics of the form "*distance* + 1/*width*" and "*distance*/*width*", respectively, where the former is the default form of the composite metric of the interior gateway routing protocol (IGRP). To the best of our knowledge, for the first time we provide shortest-path-based algorithms that are *guaranteed* to minimize these metrics in spite of their non-isotonicity. This result implies that, contrary to common belief, the composite metric of the IGRP is in fact optimizable.

1. Introduction

Routing in communication networks has been the subject of considerable interest for decades. The work of [1] and [2] introduced an algebraic theory of network routing. This work identified the necessary and sufficient conditions for any routing metric to be optimizable using shortest path (or generalized shortest path) algorithms: monotonicity and isotonicity. In particular, monotonicity of the routing metric ensures that standard shortest path algorithms (i.e., Dijkstra or Bellman-Ford) will converge, and isotonicity of the routing metric ensures that the shortest path algorithms will converge to an *optimal* path. Note that monotonicity implies that the overall weight of a path does not improve when it is extended by a new link. Isotonicity, however, implies that the weight-relationship of two paths with the same source is preserved if both routes are extended by the same link. In particular, let A and B be two paths originating from the same source, where A is shorter that B. If both paths are appended by the same link l, then isotonicity of the routing metric implies that the extended path $A \cup l$ is also shorter than $B \cup l$. For more details, refer to [2]. This algebraic theory of routing was also utilized in more recent studies, e.g., [3], [4] and [5].

Let $D(L)$ and $W(L)$ denote the length (distance) and width of path L respectively. Note that $D(L)$ is the sum of all link distances along L, and $W(L)$ is the minimum (bottleneck) link width along L. We consider finding the paths that *minimize* the following two composite metrics, respectively:

$$Met_1(L) = D(L) + \frac{1}{W(L)}, \tag{1}$$

and

$$Met_2(L) = \frac{D(L)}{W(L)}. \tag{2}$$

Note that the first metric (Met_1) is the default form of the composite metric of the interior gateway routing protocol (IGRP) [6,7]. The second metric (Met_2) is an alternative metric that attempts to achieve a tradeoff between length and width of the chosen path.

The study in [2] noted that the IGRP metric (Met_1) as given by (1) is monotone, but not isotonic. The following section will elaborate more on this fact. This implies that standard shortest path procedures (i.e., Dijkstra or Bellman-Ford) will not converge to the optimal path if used to minimize Met_1. In spite of this, the original IGRP was based on a distance-vector implementation, with the help of heuristics to prevent the formation of cycles [6,7]. Since the distance-vector protocol uses a distributed Bellman-Ford route selection algorithm, its worst-case complexity is $O(NM)$, where N and M denote the number of nodes and links, respectively. The enhanced IGRP (EIGRP) [8] uses the same routing metric Met_1 as given by (1). In fact, EIGRP is also based on a distance-vector implementation, but replaces the heuristics with the diffusing update algorithm (originally presented in [9]) to avoid the formation of loops. It has been noted in [7] that the diffusing algorithm of [9] has been originally designed for a simple distance metric, and would behave unexpectedly in the case of composite metrics such as the IGRP metric Met_1. In summary, although EIGRP is still one of

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the primary routing protocols used by Cisco Systems, its distance-vector implementation does not guarantee convergence to the optimal path due to the non-isotonicity of routing metric used. Moreover, to the best of our knowledge, shortest-path-based algorithms that are *guaranteed* to minimize the well-known IGRP (and EIGRP) metric Met_1 *exactly* have not been reported.

The alternative metric (Met_2) is not as well-studied as the IGRP metric (Met_1). Although the study in [10] focuses on multipath routing, optimizing Met_2 appears in finding an initial flow for other, more involved algorithms. In particular, Nakibly et al. [10] reports that the path with maximum $W(L)/D(L)$ can be found using dynamic programming at a running time of $O(N^2M^3\log^2(N))$, where N and M denote the number of nodes and links, respectively. Moreover, the studies in [11–13] addressed the problem of finding the path with maximum (bandwidth-normalized) end-to-end data rate in multihop wireless networks. These studies did not explicitly consider a routing metric that combines distance with width. However, the resulting routing metric can be regarded as a special case of Met_2 when time-division-multiplexing (TDM) is used among different wireless link transmissions. In short, these studies are more specific, and less general than our succinct approach.

Outside the context of isotonicity and the applicability of shortest path algorithms, studies that considered composite (or multi-objective) routing metrics do exist. For example, [14] and [15] combined several routing metrics using a multi-objective optimization approach, and proposed solutions based on meta-heuristic evolutionary algorithms. The studies in [16] and [17] also combined several routing metrics, and proposed solutions based on ant-colony optimization. Our present work, however, differs from this line of research in several aspects. In particular, these studies did not focus on the isotonicity property or the convergence of shortest path algorithms, but used general-purpose optimization methods instead. In contrast, our work focuses on the development of shortest-path-based algorithms for the non-isotonic metrics of interest, as opposed to a general-purpose optimization approach.

Given a network and a source-destination pair of nodes, this paper focuses on finding the path from source to destination such that the routing metric Met_1 (or Met_2) is minimized. Using the divide-and-conquer principle [18] from optimization theory, we divide the overall routing problem into separate sub-problems that can be solved via shortest path procedures. We also extend our approach to the case where the distance of the path $D(L)$ is replaced by its hop-count $|L|$ in the metrics. In the latter case, we will show that all resulting sub-problems can be solved using a single run of the Bellman-Ford shortest path algorithm. In particular, the contribution of this paper can be summarized as follows.

- To the best of our knowledge for the first time, we devise a *provably optimal*, shortest-path-based algorithm to solve the routing problem using the composite IGRP metric Met_1. Our approach is based on the divide-and-conquer principle, and iteratively invoking a shortest path procedure. The algorithm running time is $O(N^2M)$, where N and M denote the number of nodes and links, respectively. We demonstrate that the same algorithm (with the same $O(N^2M)$ complexity) can be applied to minimize Met_2. This is a significant improvement over the $O(N^2M^3\log^2(N))$ reported in [10].
- In the special case where the distance of the path $D(L)$ is replaced by its hop-count $|L|$ in Met_1 or Met_2, we devise an improved algorithm based on a *single* run of the Bellman-Ford algorithm at a running time of only $O(NM)$.

The remainder of this paper is organized as follows. Section 2 provides a problem definition, and an overview of the monotonicity and isotonicity of the composite routing metrics Met_1 and Met_2. Section 3 presents the shortest-path-based algorithm for minimizing Met_1 (or Met_2), and establishes its guaranteed optimality theoretically. The improved shortest-path algorithm for the case where the metrics use the hop-count instead of the distance, and its optimality proof, are presented in Section 4. Numerical results are presented in Section 5. Finally, Section 6 concludes the paper.

2. Problem definition

A network is modeled as a graph $G = (V, E)$, where V represents the set of nodes (vertices) and E represents the set of links (edges). We let $l \in E$ signify a link in the network. We also let $N = |V|$ and $M = |E|$ denote the number of nodes and links, respectively. Each link $l \in E$ is associated by a distance d_l and a width w_l. In practice, the distance of a link may be its physical distance, the delay on the link, or any additive function that reflects the cost of using the link. The width of a link typically represents its bandwidth or available bandwidth. Therefore, for any given path L in the network, we define its end-to-end distance $D(L)$ and width $W(L)$, respectively, as follows:

$$D(L) = \sum_{l \in L} d_l, \tag{3}$$

and

$$W(L) = \min_{l \in L} w_l. \tag{4}$$

Given a source-destination $s-d$ pair of nodes $(s, d) \in V \times V$, we address the following two problems:

$$\min_{L \in \mathcal{L}_{sd}} Met_1(L) = D(L) + \frac{1}{W(L)}, \tag{5}$$

and

$$\min_{L \in \mathcal{L}_{sd}} Met_2(L) = \frac{D(L)}{W(L)}, \tag{6}$$

where $\mathcal{L}_{sd}$ denotes the set of all paths joining s and d.

In what follows, we show that both routing metrics Met_1 and Met_2 are monotone, but not isotonic.

Observation 1. The routing metric Met_1 is monotone.

Proof. If path L is appended by link l, monotonicity of Met_1 implies that $Met_1(L \cup l) \geq Met_1(L)$. In other words, the overall metric does not improve if a path is appended by a link. It can be verified that

$$\begin{aligned} Met_1(L \cup l) &= D(L \cup l) + \frac{1}{W(L \cup l)} \\ &= D(L) + d_l + \frac{1}{\min\{W(L), w_l\}} \\ &\geq D(L) + d_l + \frac{1}{W(L)} \\ &\geq D(L) + \frac{1}{W(L)} \\ &= Met_1(L). \end{aligned} \tag{7}$$

Note that the second equality comes from the fact that the distance is an additive, while the width is a bottleneck quantity. The second inequality follows from the fact that link distances and widths are positive. This completes the proof. □

Observation 2. The routing metric Met_2 is monotone.

Proof. The proof is almost identical to that of Observation 1, and is omitted. □

Now, we provide an argument that both routing metrics Met_1 and Met_2 are *not* isotonic.