Journal of Computational Physics 375 (2018) 447–480

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

An entropy stable discontinuous Galerkin method for the shallow water equations on curvilinear meshes with wet/dry fronts accelerated by GPUs

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ARTICLE INFO

Article history:
Received 6 April 2018
Received in revised form 21 August 2018
Accepted 23 August 2018
Available online 5 September 2018

Keywords:
Shallow water equations
Discontinuous Galerkin spectral element method
Shock capturing
Positivity preservation
GPUs
OCCA

ABSTRACT

We extend the entropy stable high order nodal discontinuous Galerkin spectral element approximation for the non-linear two dimensional shallow water equations presented by Wintermeyer et al. (2017) [46] with a shock capturing technique and a positivity preservation capability to handle dry areas. The scheme preserves the entropy inequality, is well-balanced and works on unstructured, possibly curved, quadrilateral meshes. For the shock capturing, we introduce an artificial viscosity to the equations and prove that the numerical scheme remains entropy stable. We add a positivity preserving limiter to guarantee non-negative water heights as long as the mean water height is non-negative. We prove that non-negative mean water heights are guaranteed under a certain additional time step restriction for the entropy stable numerical interface flux. We implement the method on GPU architectures using the abstract language OCCA, a unified approach to multi-threading languages. We show that the entropy stable scheme is well suited to GPUs as the necessary extra calculations do not negatively impact the runtime up to reasonably high polynomial degrees (around $N = 7$). We provide numerical examples that challenge the shock capturing and positivity properties of our scheme to verify our theoretical findings.

1. Introduction

The shallow water equations including a non-constant bottom topography are a system of hyperbolic balance laws

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0,\\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= -ghb_x,\\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= -ghb_y,
\end{aligned}
\tag{1.1}
$$

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https://doi.org/10.1016/j.jcp.2018.08.038
0021-9991/

that are useful to model fluid flows in lakes, rivers, oceans or near coastlines, e.g. [3,28,45]. We compactly write the system (1.1) as

$$\vec{w}_t + \nabla \cdot (\vec{f}, \vec{g})^T = \vec{\mathcal{S}}, \tag{1.2}$$

with $\vec{w} = (h, hu, hv)^T$, $\vec{f} = (hu, hu^2 + \frac{1}{2}gh^2, huv)^T$ and $\vec{g} = (hv, huv, hv^2 + \frac{1}{2}gh^2)^T$ and source term $\vec{\mathcal{S}} = (0, -ghb_x, -ghb_y)^T$. The water height is denoted by $h = h(x, y, t)$ and is measured from the bottom topography $b = b(x, y)$. The total water height is therefore $H = h + b$. The fluid velocities are $u = u(x, y, t)$ and $v = v(x, y, t)$. An important steady state solution of (1.1) is the preservation of a flat lake with no velocity, the so-called "lake at rest" condition

$$\begin{aligned} &h + b = \text{const}, \\ &u = v = 0. \end{aligned} \tag{1.3}$$

A numerical scheme that preserves non-trivial steady state solutions, such as the "lake at rest" problem, is *well-balanced*, e.g. [17,32]. Methods that are not well-balanced can produce spurious waves in the magnitude of the mesh size truncation error which pollutes the solution quality. This is particularly problematic because many interesting shallow water phenomena can be interpreted as perturbations from the lake at rest condition [26].

Due to the non-linear nature of the shallow water equations (1.1), discontinuous solutions may develop regardless of the smoothness of the initial conditions. Therefore, solutions to the PDEs (1.1) are sought in the weak sense. Unfortunately, weak solutions are non-unique and additional admissibility criteria are required to extract the physically relevant solution from the family of weak solutions. One important criteria for physically relevant solutions is the second law of thermodynamics, which guarantees that the entropy of a physical system increases as the fluid evolves. In mathematics, a suitable strongly convex entropy function can be used to ensure a numerical approximation obeys the laws of thermodynamics discretely [43]. A numerical scheme that satisfies the second law of thermodynamics is said to be *entropy stable*. Due to convention, the sign of the mathematical entropy is reversed when compared to the physical entropy. While the entropy should be (nearly) conserved for smooth solutions, it must be dissipated in the presence of shocks. In order to discuss the mathematical entropy for the shallow water equations a suitable entropy function is the total energy $e = e(\vec{w})$

$$e := \frac{1}{2}h(u^2 + v^2) + \frac{1}{2}gh^2 + ghb. \tag{1.4}$$

We take the derivative of the entropy function with respect to the conservative variables $\vec{w}$ to find the set of entropy variables $\vec{q} = \frac{\partial e}{\partial \vec{w}}$, which are

$$q_1 = gH - \frac{1}{2}(u^2 + v^2), \qquad q_2 = u, \qquad q_3 = v. \tag{1.5}$$

If we contract the shallow water equations (1.1) from the left with the entropy variables and apply consistency conditions on the fluxes developed by Tadmor [42] we obtain the entropy conservation law

$$e_t + \mathcal{F}_x + \mathcal{G}_y = 0, \tag{1.6}$$

with the entropy fluxes $\mathcal{F} = \frac{1}{2}hu(u^2 + v^2) + ghu(h + b)$ and $\mathcal{G} = \frac{1}{2}hv(u^2 + v^2) + ghv(h + b)$. In the presence of discontinuities (1.6) becomes the entropy inequality

$$e_t + \mathcal{F}_x + \mathcal{G}_y \leq 0. \tag{1.7}$$

Unfortunately, for high-order numerical methods the discrete satisfaction of (1.7) does not guarantee that the approximation solution will be overshoot free, e.g. [46]. Therefore, the first contribution of this work is to add a shock capturing method that maintains the entropy stability of the nodal discontinuous Galerkin method (DGSEM) on curvilinear quadrilateral meshes developed by Wintermeyer et al. [46]. In particular, artificial viscosity is added into the two momentum equations. The amount of artificial viscosity is selected with the method developed by Persson and Peraire [33]. Even with shock capturing there can still be issues maintaining the positivity of the water height, h, particularly in flow regions where $h \to 0$. Thus, our second contribution is to incorporate the positivity preserving limiter of Xing et al. [50] in an entropy stable way. To fulfill the requirements of the positivity limiter, we formally show that the entropy stable numerical fluxes of the entropy stable discontinuous Galerkin spectral element method (ESDGSEM) preserve positive mean water heights on two-dimensional curved meshes. We then generalize a result from Ranocha [37] to show that the positivity preserving limiter itself is entropy stable on curvilinear meshes.

Our third, and final, contribution is to implement the two-dimensional positive ESDGSEM on GPUs. The entropy stable approximation is built with specific split forms, which are linear combinations of the conservative and advective forms of the shallow water equations [17,46]. However, the method remains fully conservative [11] albeit with additional computational complexity in the form of an increased number of arithmetic operations, but without the need for more data storage and transfer. Therefore, the ESDGSEM seems a perfect candidate for implementation on GPUs. We demonstrate that this