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Explicit and provably stable spatiotemporal FDTD refinement

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ABSTRACT

In this paper we introduce an explicit and provably conditionally stable Finite Difference Time Domain (FDTD) algorithm for Maxwell's equations, with local refinement in both the spatial discretization length and in the time step (spatiotemporal refinement). This enables local spatial refinement with a locally reduced time step.

1. Introduction

The Finite Difference Time Domain (FDTD) algorithm [16] is a well-known computational method for solving Maxwell's equations. It has many desirable characteristics: it is conceptually simple, easy to implement, easy to parallelize thanks to its explicit nature, and comes with strong mathematical guarantees regarding its stability and discrete conservation laws. One such guarantee is the Courant condition: FDTD is stable if the time step, Δ_t, obeys

$$c\Delta_t < \frac{\Delta}{\sqrt{d}} \tag{1}$$

where Δ is the spatial discretization length, c the speed of light and d the number of spatial dimensions. (1) is derived under the assumption of a standard Yee grid filled with a uniform medium. More general but not fundamentally different stability conditions are known in the case of non-Yee grids and non-uniform media, such as (9).

This stability condition, unfortunately, prevents us from resolving small features in the problem domain without choosing a denser grid and thus a (globally) smaller time step. This drawback has been partially overcome: there are ways of constructing spatially refined grids [1] which let us resolve small features locally, but still impose a global reduction of the time step. Features which are small in just one direction (e.g. thin layers of conducting material) can be included using partially implicit approaches, which do not impose global time step reduction nor globally denser meshes [9,20]. In fact, local spatial refinement with no global time step reduction is always possible provided one is willing to use Crank–Nicolson (implicit) updates in the refined region [19–21], or by filtering out unstable modes in the refined grid [22], but this reduces both the ease of implementation, and the ease of parallelization.

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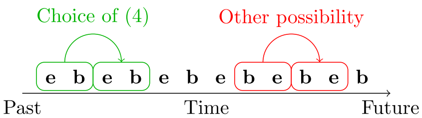

Fig. 1. The staggered nature of FDTD time-stepping, and the resulting ambiguity in the definition of the time-stepping operator. The choice of eq. (4) is the one where $[\mathbf{e}_-^T, \mathbf{b}_-^T]^T$ and $[\mathbf{e}_+^T, \mathbf{b}_+^T]^T$ are such that the electric field is discretized half a time step to the past of the magnetic field, traditionally written as $\left[\mathbf{e}^T_{(n-\frac{1}{2})\Delta_t}, \mathbf{b}^T_{n\Delta_t}\right]^T$ and $\left[\mathbf{e}^T_{(n+\frac{1}{2})\Delta_t}, \mathbf{b}^T_{(n+1)\Delta_t}\right]^T$.

In this paper we introduce an explicit spatiotemporal FDTD refinement scheme, which lets us update the refined regions with a *locally* smaller time step. We are not the first to attempt this [25,23,1,13,4,5,12,3,6], but to our knowledge this scheme is the first that is explicit, provably conditionally stable, and recursively applicable (i.e. the refined region can be further refined). In [19], we constructed an explicit 1:2 refinement scheme that is provably conditionally stable and remains stable at the coarse Courant limit, but is not recursively applicable. The schemes of [4,5] are provably conditionally stable and stable at the coarse Courant limit, but not explicit: they require, at every step, the solution of a system of equations whose size is proportional to the size of the coarse-fine interface.

This paper is structured as follows: in section 2, we derive a multirate time-stepping operator based on algebraic considerations, which we summarize as a practical algorithm in section 3. In section 4, we give a proof of its conditional stability. In section 5 we combine our multirate time-stepping operator with spatial refinement. Numerical examples are in section 6, and the conclusion is in section 7.

2. Multi-rate FDTD time-stepping: an algebraic approach

The FDTD algorithm numerically solves Maxwell's equations

$$\frac{1}{c}\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_r}\nabla \times \mu_r^{-1} B \tag{2}$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = -\nabla \times E \tag{3}$$

where E is the electric field, B the magnetic field, c the speed of light, and μ_r, ε_r the relative magnetic permeability and electric permittivity. We choose in this paper to use Gaussian units, since they make the symmetry of Maxwell's equations more readily apparent: note the factor $1/c$ in both (2) and (3).

FDTD discretizes the curl operator $\nabla\times$ on a discrete grid, with discretization length Δ, usually by using central differences to approximate the spatial derivatives. The discrete curl operator can then be represented as a matrix C.

FDTD, being a time-domain algorithm, has the task of mapping the discrete electric and magnetic fields known at the current time step, which we will call $[\mathbf{e}_-^T, \mathbf{b}_-^T]^T$, onto those at the next time step, which we will call $[\mathbf{e}_+^T, \mathbf{b}_+^T]^T$ (Due to the staggered nature of "leapfrog" FDTD time-stepping, there is a certain ambiguity in the choice of $[\mathbf{e}_-^T, \mathbf{b}_-^T]^T$ and $[\mathbf{e}_+^T, \mathbf{b}_+^T]^T$, which is resolved in Fig. 1). This mapping is achieved by solving the following equation [1,19]

$$\frac{\begin{bmatrix}\mathbf{e}_+\\ \mathbf{b}_+\end{bmatrix} - \begin{bmatrix}\mathbf{e}_-\\ \mathbf{b}_-\end{bmatrix}}{c\Delta_t} = \begin{bmatrix}0 & [\star_\epsilon]^{-1}C^T[\star_\mu^{-1}]\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}_-\\ \mathbf{b}_-\end{bmatrix} + \begin{bmatrix}0 & 0\\ -C & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}_+\\ \mathbf{b}_+\end{bmatrix} \tag{4}$$

where the l.h.s. approximates the time derivatives in Maxwell's equations, and the r.h.s. the curls. The "mass matrices" $[\star_\epsilon]$ and $[\star_\mu^{-1}]$ encode on their diagonal the (possibly inhomogeneous) relative permittivity and (inverse) magnetic permeability. (4) can be further simplified by a change of basis (from $\mathbf{e}$, $\mathbf{b}$ to $\mathcal{E}$, $\mathcal{B}$):

$$\mathcal{E} = [\star_\epsilon]^{1/2}\mathbf{e} \tag{5}$$

$$\mathcal{B} = [\star_\mu^{-1}]^{1/2}\mathbf{b} \tag{6}$$

$$\mathcal{C} = [\star_\mu^{-1}]^{1/2}\, C[\star_\epsilon]^{-1/2} \tag{7}$$

$$\frac{1}{c}\frac{\begin{bmatrix}\mathcal{E}_+\\ \mathcal{B}_+\end{bmatrix} - \begin{bmatrix}\mathcal{E}_-\\ \mathcal{B}_-\end{bmatrix}}{\Delta_t} = \underbrace{\begin{bmatrix}0 & \mathcal{C}^T\\ 0 & 0\end{bmatrix}}_{M_1}\begin{bmatrix}\mathcal{E}_-\\ \mathcal{B}_-\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0\\ -\mathcal{C} & 0\end{bmatrix}}_{M_2}\begin{bmatrix}\mathcal{E}_+\\ \mathcal{B}_+\end{bmatrix} \tag{8}$$

The anti-symmetry of (8), where $\mathcal{C}^T$ appears in the electric field update and $-\mathcal{C}$ in the magnetic field update, is sometimes called spatial reciprocity [1,24,8] and is the key ingredient in proofs that show the stability of (4) and (8). The stability condition associated with (4) and (8) is [1,19]: