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Higher-order compatible finite element schemes for the nonlinear rotating shallow water equations on the sphere

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ABSTRACT

We describe a compatible finite element discretisation for the shallow water equations on the rotating sphere, concentrating on integrating consistent upwind stabilisation into the framework. Although the prognostic variables are velocity and layer depth, the discretisation has a diagnostic potential vorticity that satisfies a stable upwinded advection equation through a Taylor–Galerkin scheme; this provides a mechanism for dissipating enstrophy at the gridscale whilst retaining optimal order consistency. We also use upwind discontinuous Galerkin schemes for the transport of layer depth. These transport schemes are incorporated into a semi-implicit formulation that is facilitated by a hybridisation method for solving the resulting mixed Helmholtz equation. We demonstrate that our discretisation achieves the expected second order convergence and provide results from some standard rotating sphere test problems.

1. Introduction

The development of new numerical discretisations based on finite element methods is being driven by the need for more flexibility in mesh geometry. The scalability bottleneck arising from the latitude–longitude grid means that weather and climate model developers are searching for numerical discretisations that are stable and accurate on pseudo-uniform grids without sacrificing properties of conservation, balance and wave propagation that are important for accurate atmosphere modelling on the scales relevant to weather and climate [31]. There is also ongoing interest in adaptively refined meshes as a way of seamlessly coupling global scale and local scale atmosphere simulations, as well as dynamic adaptivity or even moving meshes; using these meshes requires numerical methods that can remain stable and accurate on multiscale meshes. Further, there is an interest in using higher-order spaces to try to offset the inhomogeneity in the error due to using grids that break rotational symmetry.

Compatible finite element methods are a form of mixed finite element methods (meaning that different finite element spaces are used for different fields) that allow the exact representation of the standard vector calculus identities div-curl=0 and curl-grad=0. This necessitates the use of H(div) finite element spaces for velocity, such as Raviart–Thomas and Brezzi–Douglas–Marini, and discontinuous finite element spaces for pressure (stable pairing of velocity and pressure space relies on the existence of bounded commuting projections from continuous to discrete spaces, as detailed in Boffi et al. [9], for example). The main reason for choosing compatible finite element spaces is that they have a discrete Helmholtz decomposition of the velocity space; this means that there is a clean separation between divergence-free and rotational velocity fields. Cotter and Shipton [11] used this decomposition to demonstrate that compatible finite element discretisations for the linear shallow water equations on arbitrary grids satisfy the basic conservation, balance and wave propagation properties listed

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in Staniforth and Thuburn [31]. In particular, it was shown that the discretisation has a geostrophic balancing pressure for every velocity field in the divergence-free subspace of the H(div) finite element space. A survey of the stability and approximation properties of compatible finite element spaces is provided in Natale et al. [25], including a proof of the absence of spurious inertial oscillations.

The challenge of building atmosphere models using compatible finite elements is that there is no freedom to select finite element spaces in order to ensure good representation of the nonlinear equations (such as conservation, or accurate advection, for example), because the choice has already been made to satisfy linear requirements. In the case of the rotating shallow water equations, the use of discontinuous finite element spaces for the layer depth field encourages us to use upwind discontinuous Galerkin methods to solve the continuity equation describing layer depth transport.

The nonlinearity in the momentum/velocity equation is more challenging. In McRae and Cotter [24], the energy-enstrophy conserving formulation of Arakawa and Lamb [2] was extended to compatible finite element methods. This extension is closely related to C-grid methods for the shallow water equations on more general meshes in Ringler et al. [28], Thuburn and Cotter [32]. Following these approaches, the compatible finite element formulation, which has velocity and height as prognostic variables, has a diagnostic potential vorticity that satisfies a conservation equation that is implied by the prognostic dynamics for velocity and height. A finite element exterior calculus structure in this formulation was exposed in Cotter and Thuburn [12], which also provided an alternative formulation based around low-order finite element methods on dual grids. In Thuburn and Cotter [34], the close relationship of the dual grid formulation to finite volume methods was exploited to obtain a stable discretisation of the nonlinear shallow water equations on the sphere where the finite element formulation of the wave dynamics was coupled with high-order finite volume methods for the layer depth and prognostic potential vorticity fields. The essential idea is to select a particular stable accurate finite volume scheme for the diagnostic potential vorticity, and to then find the update for the prognostic velocity which implies it. In this paper we address the issue of extending this idea to higher-order finite element spaces, for which there is no analogue of the dual grid spaces. This means that we must return to the formulation of McRae and Cotter [24], where the potential vorticity is stored in a continuous finite element space. We then seek stable, accurate higher-order discretisations of the potential vorticity equation using continuous finite element methods that make it possible to find the corresponding update for prognostic velocity. It turns out that this is indeed possible for advection methods from the SUPG/Taylor–Galerkin family of methods.

Finally, we show how these discretisations can be embedded within a semi-implicit time-integration scheme. We again follow the formulation in Thuburn and Cotter [34], in which advection terms are obtained from explicit time integration methods applied using the (iterative) velocity at time level $n+1/2$. The linear system solved during each nonlinear iteration for the corrections to the field values also requires attention. The standard approach of eliminating velocity to solve a Helmholtz problem for the correction to the layer depth is problematic because the inverse velocity mass matrix is dense. We instead use a hybridised formulation where one solves for the Lagrange multipliers that enforce normal continuity of the velocity field [9, for example].

In section 2 we describe the shallow water model, including the spatial and temporal discretisation; we present finite element spaces that satisfy the properties outlined above and provide details of how to construct such spaces on the sphere and describe advection schemes for both discontinuous and continuous fields as required. In section 3 we present the results of applying our scheme to some of the standard set of test cases for simulation of the rotating shallow water equations on the sphere as described in Williamson et al. [36] and Galewsky et al. [15]. Section 4 provides a summary and brief outlook.

2. The shallow water model

2.1. Shallow water equations

We begin with the vector invariant form of the nonlinear shallow water equations on a two dimensional surface Ω embedded in three dimensions,

$$\boldsymbol{u}_t + (\zeta + f)\boldsymbol{u}^{\perp} + \nabla\left(g(D+b) + \frac{1}{2}|\boldsymbol{u}|^2\right) = 0, \tag{1}$$

$$D_t + \nabla\cdot(\boldsymbol{u}D) = 0, \tag{2}$$

where $\boldsymbol{u}$ is the horizontal velocity, D is the layer depth, b is the height of the lower boundary, g is the gravitational acceleration, f is the Coriolis parameter and $\zeta = \nabla^{\perp}\cdot\boldsymbol{u} := (\boldsymbol{k}\times\nabla)\cdot\boldsymbol{u}$ is the vorticity, $\boldsymbol{u}^{\perp} = \boldsymbol{k}\times\boldsymbol{u}$, $\boldsymbol{k}$ is the normal to the surface Ω, and where the ∇ and $\nabla\cdot$ operators are defined intrinsically on the surface. These equations have the important property that the shallow water potential vorticity (PV)

$$q = \frac{\zeta + f}{D} \tag{3}$$

satisfies a local conservation law,

$$\frac{\partial}{\partial t}(Dq) + \nabla\cdot(\boldsymbol{u}qD) = 0. \tag{4}$$