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Multiple joined prestressed orthotropic layers under large strains

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ABSTRACT

A compressible orthotropic nonlinear elastic material model is developed, for which a number of exact analytical solutions are possible for large deformations. The exact solution for the problem of bending of the compound beam with prestrained layers is obtained for large deformations using this model of elastic materials. The solution is obtained using the theory of superposition of large strains. Numerical results are shown. The nonlinear effects and the effects of anisotropy are investigated.

1. Introduction

Finding the exact analytical solutions of problems on the stress-strained state of bodies obtained by junction of prestrained parts is of particular interest in the nonlinear theory of elasticity. Such solutions can be obtained on the basis of known exact solutions of nonlinear elasticity problems (Lurie, 1990; Ogden, 1984; Rivlin, 1949a,b; Truesdell, 1972). The formulation of problems in this case is carried out based on the theory of superposition of large deformations (Levin, 1998; Levin & Taras'ev, 1983). In particular, the problems of bending a composite rectangular bar made of incompressible isotropic nonlinearly elastic materials were solved in (Levin, Zubov, & Zingerman, 2015; 2016).

Finding exact solutions for compressible materials is much more difficult than for incompressible materials, and can be obtained, as a rule, for constitutive relations (classes of materials) of a special type (De Pascalis, 2010; Fu & Ogden, 2001; Ungor, 2009). An example of such a class of isotropic compressible materials for which exact solutions can be obtained for large deformations, is a semilinear (harmonic) material, also called John's material (John, 1966).

For many materials it is important to take the anisotropy effects into account.

The constitutive equation of linear elasticity (the Hooke's law) can be generalized on the case of large strains by multiple ways. However, it is difficult to construct the constitutive equations for which exact analytical solutions are possible for large strains. Here the semilinear material model is generalized on the case of anisotropic materials. The exact analytical solution of the bending problem for a multilayer beam with prestrained layers for large deformations is obtained for this material model.

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The applications of the proposed model are related with additive technologies (Dai & Shaw, 2003; Jiang & Wang, 2016; Popovich et al., 2017; Truby & Lewis, 2016). The preliminary stretch in these technologies is caused by thermal effects. Another motivation of this research is associated with the modeling of tissue growth (Mitsuhashi, Ghosh, & Koibuchi, 2018; Rausch & Kuhl, 2013; Rodriguez, Hoger, & McCulloch, 1994). And finally, the exact analytical solutions of the problems under consideration will be useful for the verification of finite-element software that is intended for stress analysis of growing bodies with prestrained parts under large strains.

2. The orthotropic nonlinearly elastic material model

Hooke's law for orthotropic material in the linear theory of elasticity has the form (Lekhnitskii, 1963)

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E_1}\sigma_{11} - \frac{\nu_{21}}{E_2}\sigma_{22} - \frac{\nu_{31}}{E_3}\sigma_{33} \\
\varepsilon_{22} &= -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{1}{E_2}\sigma_{22} - \frac{\nu_{32}}{E_3}\sigma_{33} \\
\varepsilon_{33} &= -\frac{\nu_{13}}{E_1}\sigma_{11} - \frac{\nu_{23}}{E_2}\sigma_{22} + \frac{1}{E_3}\sigma_{33}
\end{aligned} \tag{1}$$

$$\varepsilon_{12} = \frac{1}{2G_{12}}\tau_{12}, \quad \varepsilon_{13} = \frac{1}{2G_{13}}\tau_{13}, \quad \varepsilon_{23} = \frac{1}{2G_{23}}\tau_{23} \tag{2}$$

Here ε_{sk} are linear strain tensor components in the basis of the principal axes of the material's orthotropy, σ_{ss} — normal stresses, τ_{sk} — shear stresses, G_{sk}, E_s, and ν_{sk} — elastic constants. They are related by ratios

$$E_1\nu_{21} = E_2\nu_{12}, \quad E_2\nu_{32} = E_3\nu_{23}, \quad E_3\nu_{13} = E_1\nu_{31} \tag{3}$$

It should be emphasized that relations (1), (2) are valid only in the orthonormal basis of the material's symmetry axes. A constitutive relation for an orthotropic material in an arbitrary orthonormal basis is written differently and in a more complicated way in comparison with (1) and (2).

Constitutive equations (1), (2) can be extended to the case of large deformations in different ways. The nonlinear theory of elasticity permits one to define infinitely many symmetric tensors of finite deformations coinciding with the linear strain tensor in the area of small deformations. Each of these strain tensors corresponds to the energy-conjugated stress tensor. Thus, having replaced the tensor $\boldsymbol{\varepsilon}$ in relations (1), (2) with a tensor of finite deformations, we should place the components of the corresponding energy-conjugated stress tensor instead of σ_{ss} and τ_{sk}. The model of the orthotropic nonlinearly elastic material obtained in this way will be correct, at least from the thermodynamic point of view. The purpose of further construction is to obtain a material model that allows for a number of exact solutions for large deformations. A model of semilinear (harmonic) material (John, 1966; Lurie, 1990) is distinguished by such property among isotropic compressible materials.

We replace the linear strain tensor in (1) and (2) by the tensor

$$\boldsymbol{\varepsilon} = \mathbf{U} - \mathbf{I}, \quad \mathbf{U} = \mathbf{C}^{\frac{1}{2}}, \quad \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} \tag{4}$$

where $\mathbf{I}$ – the unit tensor, $\mathbf{U}$ — symmetric positively defined right stretch tensor, $\mathbf{C}$ — left Cauchy–Green deformation tensor, $\mathbf{F}$ — deformation gradient. The symmetric Biot stress tensor $\mathbf{S}$ is energy-conjugated to the tensor $\mathbf{U}$

$$\mathbf{S} = \frac{\mathrm{d}W}{\mathrm{d}\mathbf{U}}, \quad W = W(\mathbf{U}) \tag{5}$$

Here W is the strain energy density of material (the elastic potential) (Green & Adkins, 1960). From the known (Lurie, 1990) relation

$$\overset{0}{\boldsymbol{\Sigma}} = 2\frac{\mathrm{d}W}{\mathrm{d}\mathbf{C}}, \quad \overset{0}{\boldsymbol{\Sigma}} = \mathbf{F}^{-1} \cdot \mathbf{P} \tag{6}$$

where $\overset{0}{\boldsymbol{\Sigma}}$ is the second (symmetric) Piola–Kirchhoff stress tensor, and $\mathbf{P}$ is the first (asymmetric) Piola–Kirchhoff stress tensor, it is not difficult to obtain the expression for the Biot stress tensor through the first and the second Piola–Kirchhoff stress tensors

$$\mathbf{S} = \frac{1}{2}\left(\overset{0}{\boldsymbol{\Sigma}} \cdot \mathbf{U} + \mathbf{U} \cdot \overset{0}{\boldsymbol{\Sigma}}\right) = \frac{1}{2}\left(\mathbf{P}^T \cdot \mathbf{R} + \mathbf{R}^T \cdot \mathbf{P}\right) \tag{7}$$

$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$$

Here $\mathbf{R}$ is the proper orthogonal rotation tensor (Lurie, 1990; Truesdell, 1972).

We should note that, generally speaking, symmetric tensors $\mathbf{U}$ and $\overset{0}{\boldsymbol{\Sigma}}$ do not commute, that is, $\mathbf{U} \cdot \overset{0}{\boldsymbol{\Sigma}} \neq \overset{0}{\boldsymbol{\Sigma}} \cdot \mathbf{U}$. These tensors are permutative tensors only if they are co-axial. Thus, the first and the second Piola–Kirchhoff stress tensors enter implicitly