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The concavity of p-entropy power and applications in functional inequalities

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ABSTRACT

In this paper, we prove the concavity of p-entropy power of probability densities solving the p-heat equation on closed Riemannian manifold with nonnegative Ricci curvature. As applications, we give new proofs of L^p-Euclidean Nash inequality and L^p-Euclidean Logarithmic Sobolev inequality, moreover, an improvement of L^p-Logarithmic Sobolev inequality is derived.

1. Introduction and main results

In 1948 classic paper [9], Shannon introduced the entropy power,

$$N(X) = \exp\left(\frac{2}{n} H(X)\right)$$

and gave a proof of entropy power inequality (EPI) (1.1) for independent random variables X and Y,

$$N(X+Y) \geq N(X) + N(Y), \tag{1.1}$$

where $H(X)$ was the entropy for random variable X with a probability density function u,

$$H(X) = -\int_{\mathbb{R}^n} u \log u \, dx.$$

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The entropy power inequality plays an important role in the fields of information theory, probability theory and convex geometry. There are many interesting consequences on the entropy power inequality, one of these consequences has been noticed by Costa. More precisely, in 1985, Costa [3] proved that, if $u(t)$, $t>0$, are probability densities solving the heat equation $\partial_t u = \Delta u$ in the whole space $\mathbb{R}^n$, then

$$\frac{d^2}{dt^2} N(u(t)) \leq 0. \tag{1.2}$$

Inequality (1.2) is referred to as the **concavity of entropy power**. In [11], Villani gave a direct proof of (1.2) in a strengthened version with an exact error term, which connects the concavity of entropy power with some identities of Bakry–Émery.

Recently, G. Savaré and G. Toscani [8] showed that the concavity of entropy power is a property which is not restricted to Shannon entropy power in connection with the heat equation, but it holds for the γ-th Rényi entropy power (1.3),

$$N_\gamma(u) \doteqdot \exp\left(\frac{\lambda}{n} R_\gamma(u)\right), \quad \lambda = 2 + n(\gamma-1) > 0, \tag{1.3}$$

which connects with the solution to the nonlinear diffusion equation

$$\partial_t u = \Delta u^\gamma, \tag{1.4}$$

where R_γ is the Rényi entropy

$$R_\gamma(u) \doteqdot \frac{1}{1-\gamma} \log \int_{\mathbb{R}^n} u^\gamma(x)\, dx, \quad \gamma \in (0,\infty),\ \gamma \neq 1.$$

When $\gamma > 1 - \frac{2}{n}$, they have proved the concavity of Rényi entropy power defined in (1.3),

$$\frac{d^2}{dt^2} N_\gamma(u(t)) \leq 0,$$

where $u(t), t>0$ are probability densities solving (1.4)in $\mathbb{R}^n$.

In this paper, motivated by above works, we study the entropy with respect to nonlinear diffusion on $\mathbb{R}^n$ and Riemannian manifolds. Let u be a positive solution to the p-heat equation

$$\frac{\partial u^{p-1}}{\partial t} = (p-1)^{p-1} \Delta_p u, \tag{1.5}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian of u, define the p-entropy

$$H_p(u) \doteqdot -\int_M u^{p-1} \log u^{p-1}\, dV, \quad p>1 \tag{1.6}$$

and p-entropy power

$$N_p(u) \doteqdot \exp\left(\frac{p}{n} H_p(u)\right) \tag{1.7}$$

on Riemannian manifold M, so that the Shannon's entropy and entropy power can be identified with the p-entropy and p-entropy power of index $p=2$ respectively.

Kotschwar and Ni [6] introduced p-entropy (1.6) on Riemannian manifold and proved the Perelman type W-entropy monotonicity formula with nonnegative Ricci curvature. The first author generalized this result to the weighted Riemannian manifold with nonnegative m-Bakry–Émery Ricci curvature [13] and m-Bakry–Émery Ricci curvature bounded below [12] respectively.

The first result in this paper is the concavity of p-entropy power with respect to the p-heat equation on closed Riemannian manifold with nonnegative Ricci curvature. Due to this property and motivated by