Nonlinear Analysis 179 (2019) 41–71

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Multiresolution analysis and adaptive estimation on a sphere using stereographic wavelets

Bogdan Ćmiel[a], Karol Dziedziul[b,*], Natalia Jarzębkowska[b]

[a] *Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Cracow, Poland*
[b] *Faculty of Applied Mathematics, Gdańsk University of Technology, ul. G. Narutowicza 11/12, 80-952 Gdańsk, Poland*

ARTICLE INFO

Article history:
Received 20 November 2017
Accepted 2 August 2018
Communicated by Enzo Mitidieri

MSC:
42C40

Keywords:
Smooth orthogonal projection
Sphere
Parseval frame
Adaptive estimator
Talagrand's inequality
Besov spaces
Parameter of smoothness

ABSTRACT

We construct an adaptive estimator of a density function on d dimensional unit sphere $\mathbb{S}^d$ ($d \geq 2$), using a new type of spherical frames. The frames, or as we call them, stereografic wavelets are obtained by transforming a wavelet system, namely Daubechies, using some stereographic operators. We prove that our estimator achieves an optimal rate of convergence on some Besov type class of functions by adapting to unknown smoothness. Our new construction of stereografic wavelet system gives us a multiresolution approximation of $L^2(\mathbb{S}^d)$ which can be used in many approximation and estimation problems. In this paper we also demonstrate how to implement the density estimator in $\mathbb{S}^2$ and we present a finite sample behavior of that estimator in a numerical experiment.

1. Introduction

In this paper, we consider an adaptive estimator of a density function on the d-dimensional unit sphere $\mathbb{S}^d$, $d \geq 2$ using a new type of Parseval frame. To construct the estimator we create a new stereografic wavelet system which gives us a multiresolution approximation of $L^2(\mathbb{S}^d)$. Since our construction uses a standard wavelet system (namely Daubechies) and some stereografic operators one can only make some modifications of existing algorithms in $\mathbb{R}^d$, which is relatively easy, to enjoy the benefits of multiresolution analysis on a sphere and solve many approximation and estimation problems.

Let us start from the definition

* Corresponding author.
E-mail addresses: cmielbog@mat.agh.edu.pl (B. Ćmiel), karol.dziedziul@pg.edu.pl (K. Dziedziul).

https://doi.org/10.1016/j.na.2018.08.003

Definition 1.1. Let $\{K_j : j \geq j_0\}$ be a family of measurable functions (called kernels) $K_j : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$. Let $X_1, \ldots, X_n$ be i.i.d. with density function f on $\mathbb{S}^d$ with respect to Lebesgue's measure. For $j \geq j_0$ we define an estimator of f

$$f_n(j)(x) = \frac{1}{n}\sum_{i=1}^{n} K_j(x, X_i).$$

We denote the balls of density functions in Besov spaces on sphere

$$\Sigma(s, \tilde{B}) = \{f \in B^s_{2,\infty}(\mathbb{S}^d) : \int_{\mathbb{S}^d} f(x)d\sigma_d(x) = 1,\ f \geq 0,\ \|f\|_{s,2} \leq \tilde{B}\}.$$

Since we want to obtain an adaptive estimator, we want to construct kernels K_j on a sphere for which we have an optimal rate of estimation. Namely,

Theorem 1.1. *Let $d/2 < r < R$ and let $X_1, \ldots, X_n$ be i.i.d. with density function $f \in B^s_{2,\infty}(\mathbb{S}^d)$, where s is unknown and $r \leq s \leq R$. Then there is a family of kernels $\{K_j : j \geq j_0\}$ such that for any $U > 0$ there are constants $c = c(r, R, U)$ and $C = C(U)$ such that for all s, n and $\tilde{B} > 1$ we have*

$$\sup_{f \in \Sigma(s,\tilde{B}), \|f\|_\infty \leq U} \mathbb{E}\|f_n(j_n) - f\|_2^2 \leq c\tilde{B}^{2d/(2s+d)} n^{-2s/(2s+d)},$$

where

$$j_n = \min\left\{ j \in [j_{min}, j_{max}] : \forall_{l,\ j<l\leq j_{max}}\ \ \|f_n(j) - f_n(l)\|_2^2 \leq C\frac{2^{ld}}{n} \right\}$$

and

$$j_{min} = \left\lfloor \frac{\log_2 n}{2R + d} \right\rfloor, \qquad j_{max} = \left\lceil \frac{\log_2 n}{2r + d} \right\rceil.$$

In the above theorem the smoothness parameter s is unknown but for choosing the resolution level we use a lower bound r and an upper bound R. Let us discuss some consequences of choosing different values for r and R. It seems that it is good idea to take r as small as possible and R as big as possible to consider a very wide range for the unknown smoothness. The first part of that is true since there are no serious consequences of taking small r. Unfortunately if we take a big value for R, then we need to use in our construction some very smooth wavelets (smoother than R). The smoother the wavelets are, the bigger support they have and if one scales them to a smaller area, then they change values very rapidly. In the asymptotic point of view this is not a problem but for fixed n the estimator loses its efficiency if the value R is too big. The same problem we can observe in case of a wavelet estimation on $\mathbb{R}$.

It is well-known (see Hall, Kerkyacharian and Picard (1998) [18] Theorem 4.1) that on the real line if one considers wavelets estimators with a block thresholding procedure, one attains minimax rate of convergence without extraneous logarithmic factors for $B^s_{2,\infty}$ Besov spaces and L^2-loss, i.e., $n^{-2s/(1+2s)}$. Similar result was given in [6]. We follow the arguments presented there.

The problem of estimating nonparametrically a density on the d-dimensional unit sphere $\mathbb{S}^d$ over Besov classes is not new (see Baldi, Kerkyacharian, Marinucci and Picard (2009) [2] for a direct setting and for an indirect setting see Kerkyacharian, Pham Ngoc and Picard (2011), [22]). In particular, in Baldi, Kerkyacharian, Marinucci and Picard (2009), the authors had already dealt with the considered problem in a more general framework, namely, by considering $B^s_{q,r}$ Besov spaces. They constructed an adaptive estimator based on a set of spherical wavelets, named needlets, with a hard thresholding procedure. They obtained minimax rates of convergence for $B^s_{q,r}$ Besov spaces, L^p-loss and sup-norm loss up to a logarithmic