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Interpolations and fractional Sobolev spaces in Carnot groups

Ali Maalaoui[a], Andrea Pinamonti[b,*]

[a] *Department of Mathematics and Natural Sciences, American University of Ras Al Khaimah, PO Box 10021, Ras Al Khaimah, United Arab Emirates*
[b] *Dipartimento di Matematica, Universita di Trento, Via Sommarive 14, 38123 Povo, Trento, Italy*

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ABSTRACT

In this paper we present an interpolation approach to the fractional Sobolev spaces in Carnot groups using the K-method. This approach provides us with a different characterization of these Sobolev spaces, moreover, it provides us with the limiting behavior of the fractional Sobolev norms at the end-points. This allows us to deduce results similar to the Bourgain–Brezis–Mironescu and Maz'ya–Shaposhnikova in the case $p > 1$ and Dávila's result in the case $p = 1$. Also, this allows us to deduce the limiting behavior of the fractional perimeter in Carnot groups.

1. Introduction

Carnot groups appear as the first level extension of the classical Euclidean spaces, in the sense that they are modeled over $\mathbb{R}^n$ but with a different group structure. Nevertheless, they share many analytical properties with the Euclidean case. The typical example of Carnot group is the classical Heisenberg group. Lately, there has been a lot of interest in PDEs and fractional PDEs in this group coming from a geometric background since it is the flat context of CR-geometry, see for instance [23,22,34,26,27] and the references therein. Moreover, Carnot groups have also been largely studied in several aspects, such as differential geometry [13], subelliptic differential equations [5,18,17,36] and complex variables [39]. For a general introduction to Carnot groups from the point of view of the present paper and for further examples, we refer, e.g., to [5,18,39].

It is natural then to investigate to which extent one can generalize to Carnot groups the analytical tools that are well understood in the Euclidean case, see for instance [16,28].

In this setting, we propose to study fractional Sobolev spaces from an interpolation point of view. Fractional Sobolev spaces in the literature, are also called Aronszajn, Gagliardo or Slobodeckij spaces, by the name of the ones who introduced them, almost simultaneously [2,24,37]. In Carnot groups fractional Sobolev spaces have been introduced and studied in [18,17] and many different characterizations are now present, such as the ones in [35]. In the present paper we use the K-method for real interpolation, see for instance [4],

* Corresponding author.
E-mail addresses: ali.maalaoui@aurak.ac.ae (A. Maalaoui), Andrea.Pinamonti@unitn.it (A. Pinamonti).

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to give an alternative characterization of fractional Sobolev spaces in Carnot groups. As a consequence, we derive a Bourgain–Brezis–Mironescu [6–8] (Theorem 5.1) and Ma'zya–Shaposhnikova type limiting behavior (Theorem 5.2) of the Sobolev norms similarly to the approach developed in [29]. We point out that the exact limit of the fractional Sobolev norm (as the fractional parameter goes to 1) was investigated in [3] using exact and technical computations. We also bring to the reader's attention the extensions of these types of results to other settings and to different functionals as in [1,8,9,11,10,30–32,38]. For $p = 1$ we provide a limiting behavior leading to the space of BV-functions that are of great interest in geometric measure theory in the setting of Carnot groups, see [19,21,20]. This will allow us to characterize the fractional perimeter in Carnot groups and understand its limiting behavior when the fractional parameter goes to 1, as it was done in the Euclidean setting in [15,33].

This manuscript is structured as follows: First, in Section 2, we present the structure of Carnot groups and define Sobolev Spaces and BV-Spaces in this setting. In Section 3, we provide the necessary notations, definitions and properties of the K-interpolation, which will be the main tool in our investigation. In Section 4, we provide another characterization of the K function in Carnot groups. This allows us to deduce an alternative characterization of the fractional Sobolev spaces. Finally, in Section 5, we provide applications of the characterizations given in Section 3. Namely, we present the limiting behavior of the Fractional Sobolev norms in the two end points, allowing us to obtain results similar to the ones already proved by Bourgain–Brezis–Mironescu and by Ma'zya–Shaposhnikova in [6–8] for the case $p > 1$ and by Dávila in [15] for the case $p = 1$. Also, we provide an alternative definition and characterization to the fractional perimeter and its limiting behavior at the end-points as in [15,33].

2. Carnot groups

A connected and simply connected stratified nilpotent Lie group $(\mathbb{G}, \cdot)$ is said to be a *Carnot group of step k* if its Lie algebra $\mathfrak{g}$ admits a *step k stratification*, i.e., there exist linear subspaces $V_1, \dots, V_k$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k, \tag{2.1}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$.

Set $m_i = \dim(V_i)$, for $i = 1, \dots, k$ and $h_i = m_1 + \cdots + m_i$, so that $h_k = n$. For sake of simplicity, we write also $h_0 = 0$, $m := m_1$. We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e., we set

$$Q := \sum_{i=1}^{k} i \dim(V_i).$$

We choose now a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ adapted to the stratification of $\mathfrak{g}$, i.e., such that $e_{h_{j-1}+1}, \dots, e_{h_j}$ is a basis of V_j for each $j = 1, \dots, k$. Moreover, let $X = \{X_1, \dots, X_n\}$ be the family of left invariant vector fields such that $X_i(0) = e_i$, $i = 1, \dots, n$. The exponential mapping $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism. Given a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ adapted to the stratification, any $x \in \mathbb{G}$ can be written in a unique way as

$$x = \exp(x_1 X_1 + \cdots + x_n X_n) = e^{x_1 X_1 + \cdots + x_n X_n}.$$

We identify x with $(x_1, \dots, x_n) \in \mathbb{R}^n$ and hence $\mathbb{G}$ with $\mathbb{R}^n$. This is known as *exponential coordinates of the first kind*.

The sub-bundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \dots, X_m$ is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H_x\mathbb{G} = \operatorname{span}\{X_1(x), \dots, X_m(x)\}, \qquad x \in \mathbb{G}.$$