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Measure-valued solutions to the Ericksen–Leslie model equipped with the Oseen–Frank energy☆

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ABSTRACT

In this article, we prove the existence of measure-valued solutions to the Ericksen–Leslie system equipped with the Oseen–Frank energy. We introduce a suitable concept of generalized gradient Young measures for this system. Via a Galerkin approximation, we show the existence of weak solutions to a regularized system and attain measure-valued solutions for vanishing regularization. Additionally, it is shown that the measure-valued solution fulfills an energy inequality.

1. Introduction

Nonlinear partial differential equations require generalized solution concepts. The nowadays widely accepted generalization in the form of weak solutions relies on relaxing the point-wise classical formulation into an "weaker" notion by integrating the product with a test function and applying integration-by-parts formulas. While this mechanism was first used by Lagrange [37], the associated weak convergence in Lebesgue spaces to attain a limit was observed by Wiener [64] and more prominently by Leray [40] in the context of Navier–Stokes equations (see also [49] and [13]). A further step in the generalization of solution concepts by replacing the function by a parametrized measure, a so-called Young measure, leading to measure-valued solutions was first introduced by Tartar [62]. Later on, the concept of *generalized Young measures* was used by DiPerna and Majda [14] to define generalized solutions to the Euler equations. These generalized Young measures capture oscillation and concentration effects for sequences bounded in L^1. Alibert and

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Bouchitté [2] observed that concentrations can only occur almost everywhere. Further investigations are due to Kružik and Roubíček who manage to fully characterize the set of generalized Young measures and their attainability and geometric properties [35,36]. A full treatment of Young measures and its generalizations can be found in the monograph [57] by Roubíček. We also want to mention the article by Brenier, De Lellis and Székelyhidi [6] showing the weak–strong uniqueness of measure-valued solutions to the Euler equation, because the techniques introduced there can be transferred to the setting presented here to show additional properties of the limiting measures, as well as the weak–strong uniqueness in [38].

In the article at hand, we further generalize these concepts to prove global existence of *measure-valued solutions* to the *Ericksen–Leslie system* describing *nematic liquid crystal flow*.

Nematic liquid crystals are anisotropic fluids. They consist of rod-like molecules that build or are dispersed in a fluid and are directionally ordered. This ordering and its direction heavily influence the properties of the material such as light scattering or flow behavior. This gives rise to many applications, among which *liquid crystal displays* are only the most prominent one. The Ericksen–Leslie model is a generally accepted model to describe nematic liquid crystals. The direction of the aligned molecules is modeled by a unit-vector field and the fluid flow by a velocity field. Since this model has been proposed in the 60s by Ericksen [19] and Leslie [41], it has been extensively studied. Nevertheless, the global mathematical existence theory was restricted to simple quadratic free energies that do not account for the anisotropic elastic properties of the material.

The first mathematical analysis of a simplified Ericksen–Leslie model is due to Lin and Liu [42]. They show global existence of weak solutions and local existence of strong solutions. Additionally, they manage to generalize these results to a more realistic model [44]. They also show partial regularity of weak solutions to the considered system [43]. Following this work, there have been many articles considering slightly more complicated models, for example [4,8], or [22]. Nevertheless to the best of the author's knowledge, the only generalization with respect to the free energy potential is performed by Emmrich and the author in [18]. There are also results on the *local existence of solutions* to the full Ericksen–Leslie model, see [31,63], or [30]. Especially, local strong solutions are known to exist to different simplifications of the system considered in this article. The full (thermodynamically consistent) Ericksen–Leslie system equipped with the Dirichlet energy is considered in [30], whereas the simplified Ericksen–Leslie system with the full Oseen–Frank energy is studied in [31] as well as in [32].

Already Leslie suggests to equip the Ericksen–Leslie model with the Oseen–Frank energy. It can be seen as the physically most relevant free energy function. Its simplification, the one-constant approximation, considered in the literature so far does not account for the anisotropic elastic properties of the material (see [10, Section 3.1.2]). Therefore, a global existence theory seems to be of great interest to gain further mathematical insight into the model and possibly approximate solutions to it in a decent manner.

But, we do not aim to conceal that measure-valued solutions have several disadvantages. One being that the nine-dimensional gradient of the three-dimensional function is replaced by an infinity-dimensional parametrized measure. This explosive growth of the degrees of freedom can lead to a dramatic loss of selectivity [60], which in some cases lacks any physical interpretation, and uniqueness seems to be out of reach.

Nevertheless, the concept seems to make sense for the considered model since it arises from several shortcomings of the model. For instance, the nematic phase that is modeled is not stable in the sense that a nematic material in equilibrium subjected to shearflow can evolve a biaxial character, *i.e.* two predominant directions (see [54]). Effects due to phase transition possibly influence the structure of the liquid crystal [11]. In numerical studies different instabilities occur [56], for instance tumbling of the director may be described by an oscillation measure. Additionally, it is well known that nematic liquid crystals develop singularities [1], which has been observed analytically [33] and experimentally [26]. From a mathematical point of view it is worth noticing that the Oseen–Frank energy is not convex [17], which deprives us from using standard