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TREND TO EQUILIBRIUM OF RENORMALIZED SOLUTIONS TO REACTION-CROSS-DIFFUSION SYSTEMS

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Abstract. The convergence to equilibrium of renormalized solutions to reaction-cross-diffusion systems in a bounded domain under no-flux boundary conditions is studied. The reactions model complex balanced chemical reaction networks coming from mass-action kinetics and thus do not obey any growth condition, while the diffusion matrix is of cross-diffusion type and hence nondiagonal and neither symmetric nor positive semi-definite, but the system admits a formal gradient-flow or entropy structure. The diffusion term generalizes the population model of Shigesada, Kawasaki and Teramoto to an arbitrary number of species. By showing that any renormalized solution satisfies the conservation of masses and a weak entropy-entropy production inequality, it can be proved under the assumption of no boundary equilibria that *all* renormalized solutions converge exponentially to the complex balanced equilibrium with a rate which is explicit up to a finite dimensional inequality.

1. Introduction

Multi-species systems appear in many applications in biology, physics and chemistry, and can be modeled by reaction-cross-diffusion systems. We want to study the convergence to equilibrium of reaction-cross-diffusion systems with strongly growing reactions, where the system (without reactions) is of formal gradient-flow structure and thus admits an entropy estimate. But since the reactions do not obey any growth condition, this estimate is not enough to define weak solutions, which motivates the study of renormalized solutions *à la* J. Fischer [15]. Our goal is to show that any renormalized solution satisfies the conservation of masses and a weak entropy-entropy production inequality, and consequently, under the assumption of no boundary equilibria, all renormalized solutions converge to equilibrium with an exponential rate which is explicit up to a finite dimensional inequality.

The convergence to equilibrium for reaction-diffusion systems with linear diffusion has been studied extensively, see e.g. [1, 9, 11] and references therein, while much less is known for nonlinear diffusion or cross diffusion, see [17] for a porous-medium type diffusion and [8] for Maxwell-Stefan diffusion. In this work, we study the convergence to equilibrium for a cross-diffusion model originally introduced by Shigesada, Kawasaki and Teramoto [21] in population dynamics. The existence of global weak solutions for this class of cross-diffusion models with at most linearly growing reactions has been attracted a lot of attention recently by exploiting its formal gradient-flow structure, see *e.g* [2, 3, 4, 12, 13, 19, 20]. Unfortunately, for strongly growing reactions (such as chemical reactions) this does not provide enough regularity to define weak solutions. Hence, the notion of renormalized solutions was introduced in [5] for reaction-cross-diffusion systems in analogy to [15] for reaction-diffusion systems. The standard way for proving convergence to equilibrium via entropy method is to first prove the convergence for an approximate solution, and then by passing to the limit to obtain it also for the *constructed* weak solution (see *e.g.* [8]). But since uniqueness for cross diffusion is a very delicate topic (see *e.g.* [6]), it is desirable to prove convergence to equilibrium for *all* solutions. This has been recently obtained in [18] for reaction-diffusion systems, and thus in this work, we extend these results to reaction-cross-diffusion systems with strongly growing complex balanced reactions coming from mass-action kinetics.

More precisely, we consider n chemical substances $S_1, \dots, S_n$ reacting via R reactions of the form

$$y_{r,1}S_1 + \dots + y_{r,n}S_n \xrightarrow{k_r} y'_{r,1}S_1 + \dots + y'_{r,n}S_n \qquad \text{or shortly} \qquad y_r \xrightarrow{k_r} y'_r, \qquad r = 1, \dots, R, \tag{1}$$

where $y_r = (y_{r,1}, \dots, y_{r,n}), y'_r = (y'_{r,1}, \dots, y'_{r,n}) \in (\{0\} \cup [1, \infty))^n$ are the stoichiometric coefficients, and $k_r > 0$ are the reaction rate constants. The corresponding reaction-cross-diffusion system reads for each $i = 1, \dots, n$ as

$$\begin{cases} \partial_t u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right) = f_i(u), & \text{for } (x,t) \in \Omega \times (0,T), \\ \left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right) \cdot \nu = 0, & \text{for } (x,t) \in \partial\Omega \times (0,T), \\ u_i(x,0) = u_{i,0}(x), & \text{for } x \in \Omega, \end{cases} \tag{S}$$

where $u = (u_1, \dots, u_n)$ are the population densities and Ω is a bounded domain with smooth boundary $\partial\Omega$, and ν is the exterior unit normal vector to $\partial\Omega$. The reaction terms represent the reactions in (1), i.e.

$$f_i(u) = \sum_{r=1}^R k_r(y'_{r,i} - y_{r,i})u^{y_r} \qquad \text{with} \qquad u^{y_r} = \prod_{i=1}^n u_i^{y_{r,i}}, \tag{2}$$

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