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ON NONEXISTENCE OF KNESER SOLUTIONS OF THIRD-ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to complement existing oscillation results for third-order neutral delay differential equations by establishing sufficient conditions for nonexistence of so-called Kneser solutions. Combining newly obtained results with existing ones, we attain oscillation of all solutions of the studied equations.

1. Introduction

Consider the third-order neutral delay differential equation of the form

$$\left(r_2\left(r_1 y'\right)'\right)'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $y(t) := x(t) + p(t)x(\tau(t))$. Throughout the paper, we will assume that

(H$_1$) σ, $\tau \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$, $\sigma(t) < t$, $\tau' \geq \tau_0 > 0$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$;

(H$_2$) p, $q \in \mathcal{C}([t_0,\infty),[0,\infty))$, $0 \leq p(t) \leq p_0 < \infty$ and q does not vanish identically;

(H$_3$) r_1, $r_2 \in \mathcal{C}([t_0,\infty),(0,\infty))$ satisfy $\int^\infty r_1^{-1}(t)\mathrm{d}t = \int^\infty r_2^{-1}(t)\mathrm{d}t = \infty$;

and either

(H$_{4a}$) $\tau(t) \leq t$ and $\tau\circ\sigma = \sigma\circ\tau$;

or

(H$_{4b}$) $\tau(t) > t$, $\sigma' > 0$ and $\left(\sigma^{-1}(t)\right)' \geq \sigma_0 > 0$.

For the sake of brevity, we define the operators

$$L_0 y = y, \quad L_1 y = r_1 y', \quad L_2 y = r_2\left(r_1 y'\right)', \quad L_3 y = \left(r_2\left(r_1 y'\right)'\right)'.$$

By a solution to equation (1.1), we mean a nontrivial function $x \in \mathcal{C}([T_x,\infty),\mathbb{R})$ with $T_x \geq t_0$, which has the property y, $L_1 y$, $L_2 y \in \mathcal{C}^1([T_x,\infty),\mathbb{R})$, and satisfies (1.1) on $[T_x,\infty)$. We only consider those solutions of (1.1) which exist on some half-line $[T_x,\infty)$ and satisfy the condition $\sup\{|x(t)| : T \leq t < \infty\} > 0$ for any $T \geq T_x$.

As is customary, a solution x of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory*. The equation itself is termed *oscillatory* if all its solutions oscillate.

Following classical results of Kiguradze and Kondrat'ev (see, e.g., [8]), we say that (1.1) has *property A* if any solution x of (1.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Instead of using property A, some authors say that equation is *almost oscillatory*.

To start with, let us state a characterization of possible nonoscillatory, solutions of (1.1). The following result is a modification of the well known Kiguradze lemma [8, Lemma 1.1] based on (H$_3$).

Lemma 1. *Assume* (H$_1$)$-$(H$_3$) *and x is a nonoscillatory solution of* (1.1). *Then there are only two possible classes for y:*

$$N_0 = \{y(t) : (\exists T \geq t_0)(\forall t \geq T)\,(y(t)L_1 y(t) < 0,\; y(t)L_2 y(t) > 0)\}$$
$$N_2 = \{y(t) : (\exists T \geq t_0)(\forall t \geq T)\,(y(t)L_1 y(t) > 0,\; y(t)L_2 y(t) > 0)\}.$$

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