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Generalised Seiberg–Witten equations and almost-Hermitian geometry

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ABSTRACT

In this article, we study a generalisation of the Seiberg–Witten equations, replacing the spinor representation with a hyperKähler manifold equipped with certain symmetries. Central to this is the construction of a (non-linear) Dirac operator acting on the sections of the non-linear fibre-bundle. For hyperKähler manifolds admitting a hyperKähler potential, we derive a transformation formula for the Dirac operator under the conformal change of metric on the base manifold.

As an application, we show that when the hyperKähler manifold is of dimension four, then, away from a singular set, the equations can be expressed as a second order PDE in terms of almost-complex structure on the base manifold, and a conformal factor. This extends a result of Donaldson to generalised Seiberg–Witten equations.

1. Introduction

Let X be a 4-dimensional, oriented, smooth, Riemannian manifold and let $Q \to X$ be a Spin-structure. A spinor bundle over X is a vector bundle associated to Q, with typical fibre $\mathbb{H}$. The idea for generalisation is to replace the spinor representation with a hyperKähler manifold (M, g_M, I_1, I_2, I_3) equipped with an isometric action of Sp(1) (or SO(3)) which *permutes* the complex structures on M. We will often refer to M as the *target hyperKähler manifold*. The sections of the non-linear fibre-bundle now play the role of spinors. The interplay between the Sp(1) (or SO(3)) action and the quaternionic structure on M allows one to define the Clifford multiplication. Composing the Clifford multiplication with the covariant derivative gives the generalised Dirac operator, which we denote by $\mathcal{D}$.

In order to define a generalisation of the Seiberg–Witten equations, we need additionally a twisting principal G-bundle $P_G \to X$, with a tri-Hamiltonian action of G on M. The action gives rise to a hyperKähler moment map $\mu : M \longrightarrow \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$. For a connection A on P_G and a spinor u, the 4-dimensional generalised Seiberg–Witten equations on X are the following system of equations

$$\begin{cases} \mathcal{D}_A u = 0 \\ F_A^+ - \mu \circ u = 0 \end{cases} \tag{1}$$

where $\mathcal{D}_A$ is a twisted Dirac operator for a connection A on P_G.

This non-linear generalisation of the Dirac operator is well-known to physicists and has been used in the study of gauged, non-linear σ-models [1]. The 3-dimensional version of Eqs. (1) was studied by Taubes [2] (see also [3]). The

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4-dimensional generalisation was considered by Pidstrygach [4], Schumacher [5] and Haydys [6]. The moduli spaces of solutions to (1) makes for an interesting study, especially because of its application to gauge theories on manifolds with special holonomies (cf. [7,8]). Many well-known gauge-theoretic equations like the PU(2)-monopole equations [9], the Vafa–Witten equations [10], Pin(2)-monopole equations [11], the non-Abelian monopole equations [12], etc. can be treated as special cases of this generalisation.

It is possible to obtain the target hyperKähler manifold with requisite symmetries from *Swann's construction* [13,14]. Starting with a quaternionic Kähler manifold N of positive scalar curvature, Swann constructs a fibration $\mathcal{U}(N) \to N$, whose total space admits a hyperKähler structure. Such manifolds are characterised by the existence of a hyperKähler potential. Alternatively, the permuting Sp(1)-action extends to a homothetic action of $\mathbb{H}^*$. The bundle construction commutes with the hyperKähler quotient construction of Hitchin, Karlhede, Lindström and Roček [15] and the quaternionic Kähler quotient construction of Galicki and Lawson [16]. As a result, many examples of (finite dimensional) hyperKähler manifolds with homothetic $\mathbb{H}^*$-action can be obtained via hyperKähler reduction of $\mathbb{H}^n$.

With $M = \mathcal{U}(N)$, we derive a transformation formula for the generalised Dirac operator, under the conformal change of metric on the base manifold. Since $\mathcal{U}(N)$ admits a natural homothetic action of $\mathbb{R}^+$, this setting allows one to make sense of "weighted spinors".

Let $\pi_1 : P_{\mathrm{CO}(4)} \to X$ be the bundle of conformal frames with respect to the conformal class $[g_X]$ and $P_G \to X$ be a principal G-bundle over X. Assume that the action of G on M is tri-Hamiltonian. Let $\widetilde{\pi} : \widetilde{Q} \to X$ denote the conformal $\mathrm{Spin}^G(4)$-bundle, which is a double cover of $P_{\mathrm{CO}(4)} \times_X P_G$.

Theorem 1.1. *Let f be a smooth, real-valued function on X and let u be a (generalised) spinor. Consider the metric $g'_X := e^{2f} g_X$ in the conformal class $[g_X]$ and let φ' and φ be the Levi-Civita connections associated to g_X and g'_X respectively. For a fixed connection A on P_G, denote by A_φ and $A_{\varphi'}$ the corresponding lifts to $\widetilde{Q}$. Then, the associated generalised Dirac operators $\mathcal{D}_{A_\varphi}$ and $\mathcal{D}_{A_{\varphi'}}$ are related as*

$$\mathcal{D}_{A_{\varphi'}}(\mathscr{B}u) = \mathscr{B}\left(de^{-5/2\pi_1^* f}\mathcal{D}_{A_\varphi}(e^{3/2\pi_1^* f}u)\right) \tag{2}$$

where, $\mathscr{B}$ is the lift of the automorphism $B : P_{\mathrm{CO}(4)} \longrightarrow P_{\mathrm{CO}(4)}$, given by $p \longmapsto e^{-f}p$, and $de^{-5/2\pi_1^* f}$ is the action of $e^{-5/2\pi_1^* f}$ by differential on TM.

For $M = \mathbb{H}$, the result was proved by Hitchin [17].

Assume that $M = \mathcal{U}(N)$ is a 4-dimensional hyperKähler manifold. Using the above theorem, we show that away from a singular set, the generalised Seiberg–Witten equations can be interpreted in terms of almost-complex geometry of the underlying 4-manifold, as equations for a compatible almost-complex structure and a real-valued function which is associated to a conformal factor. Recall that on a Riemannian 4-manifold (X, g_X), the compatible almost-complex structures on X are parameterized by sections of the twistor bundle $\mathcal{Z}$, which is a sphere bundle in Λ^+. Thus the almost-complex structures can be thought of as self-dual, 2-forms Ω with $|\Omega| = 1$. An almost-complex structure gives a splitting of Λ^+ into the direct sum of the trivial bundle spanned by Ω and its orthogonal complement $\overline{K}$, where K is a complex line bundle. Since $|\Omega| = 1$, its covariant derivative is a section of $T^*X \otimes_{\mathbb{R}} \overline{K}$. Using the almost-complex structure, we get the isomorphism

$$T^*X \otimes_{\mathbb{R}} \overline{K} \cong T^*X \otimes_{\mathbb{C}} K \oplus T^*X \otimes_{\mathbb{C}} \overline{K}.$$

Moreover, the wedge product gives a complex, bi-linear map

$$T^*X \times T^*X \longrightarrow \Lambda^2 T^*X = K.$$

using which, we can identify $TX \cong T^*X \otimes_{\mathbb{C}} \overline{K}$. Thus $\nabla\Omega$ has two components: the first component in $T^*X \otimes_{\mathbb{C}} K$ is the Nijenhuis tensor and the second one in TX is $d\Omega$. Let $\langle\cdot,\cdot\rangle$ denote the obvious $\overline{K}$-valued pairing between TX and $T^*X \otimes \overline{K}$.

Let $G = \mathrm{U}(1)$ and $M = \mathcal{U}(N)$ be 4-dimensional hyperKähler manifold, which is total space of a Swann bundle, equipped with a tri-Hamiltonian action of U(1) that commutes with the permuting Sp(1)-action. We will call such an action a permuting action of $\mathrm{U}(2) \cong \mathrm{Sp}(1) \times_{\pm} \mathrm{U}(1)$.

Theorem 1.2. *Fix a metric g_X on X and let $[g_X]$ be its conformal class. Assume that M is obtained as a quotient of a flat, quaternionic space and equipped with a residual permuting action of* U(2) *from the flat space. Then, there exists a bijective correspondence between the following:*

- *pairs consisting of a metric $g'_X \in [g_X]$ and a solution (u, A) to the generalised Seiberg–Witten equations, such that the image of u does not contain a fixed point of the* U(1) *action on M*
- *pairs consisting of a metric $g''_X \in [g_X]$ and a self-dual 2-form Ω satisfying*

$$(\nabla^*\nabla\Omega)^\perp + 2\langle d\Omega, N_\Omega\rangle = 0, \quad \frac{3}{2}|N_\Omega|^2 + \frac{1}{2}|d\Omega|^2 + \frac{1}{2}s_X(g''_X) < 0 \tag{3}$$

where $s_X(g''_X)$ denotes the scalar curvature with respect to the metric g''_X.