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Quasi-elliptic cohomology I

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ABSTRACT

Quasi-elliptic cohomology is a variant of elliptic cohomology theories. It is the orbifold K-theory of a space of constant loops. For global quotient orbifolds, it can be expressed in terms of equivariant K-theories. Thus, the constructions on it can be made in a neat way. This theory reflects the geometric nature of the Tate curve. In this paper we provide a systematic introduction of its construction and definition.

1. Introduction

An elliptic cohomology theory is an even periodic multiplicative generalized cohomology theory whose associated formal group is the formal completion of an elliptic curve. The elliptic cohomology theories form a sheaf of cohomology theories over the moduli stack of elliptic curves $\mathcal{M}_{ell}$. Tate K-theory over $\mathrm{Spec}\mathbb{Z}((q))$ is obtained when we restrict it to a punctured completed neighborhood of the cusp at ∞, i.e. the Tate curve $Tate(q)$ over $\mathrm{Spec}\mathbb{Z}((q))$ [Section 2.6, [2]]. The relation between Tate K-theory and string theory

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is better understood than most known elliptic cohomology theories. In addition, Tate K-theory has the closest ties to Witten's original insight that the elliptic cohomology of a space X is related to the $\mathbb{T}$-equivariant K-theory of the free loop space $LX = \mathbb{C}^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops. Ganter gave a careful interpretation in Section 2, [6] of this statement that the definition of G-equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold.

Other than the theory over $\mathrm{Spec}\mathbb{Z}((q))$, we can define variants of Tate K-theory over $\mathrm{Spec}\mathbb{Z}[q]$ and $\mathrm{Spec}\mathbb{Z}[q^\pm]$ respectively. The theory over $\mathrm{Spec}\mathbb{Z}[q^\pm]$ is of especial interest. Inverting q allows us to define a sufficiently non-naive equivariant cohomology theory and to interpret some constructions more easily in terms of extensions of groups over the circle. The resulting cohomology theory is called quasi-elliptic cohomology. Its relation with Tate K-theory is

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X) \tag{1.1}$$

which also reflects the geometric nature of the Tate curve. As discussed in Remark 3.20, $QEll^*_{\mathbb{T}}(\mathrm{pt})$ has a direct interpretation in terms of the Katz–Mazur group scheme T [Section 8.7, [14]]. The idea of quasi-elliptic cohomology is motivated by Ganter's construction of Tate K-theory [5]. It is not an elliptic cohomology but a more robust and algebraically simpler treatment of Tate K-theory. This new theory can be interpreted in a neat form by equivariant K-theories. Some formulations in it can be generalized to equivariant cohomology theories other than Tate K-theory.

Via quasi-elliptic cohomology theory, we show in this paper that G-equivariant Tate K-theory for any compact Lie group G is given by the $\mathbb{T}$-equivariant K-theory of the ghost loops [Section 2.4], or constant loops [Section 2.3] inside the free loop space LX. Moreover, as shown in Section 4.1, quasi-elliptic cohomology can be defined not only for G-spaces but also for orbifolds. Applying the same idea, we obtain a loop construction for orbifold Tate K-theory via orbifold quasi-elliptic cohomology theory.

This paper aims to provide a reference for this elegant theory and a systematic introduction of its construction and definition. In Section 2, for any compact Lie group G, we construct G-equivariant quasi-elliptic cohomology from a loop space via bibundles. Thus, we in fact give a construction by loop space of G-equivariant Tate K-theory for compact Lie groups G. In Section 2 [11] we showed the construction when G is a finite group, which, as shown in Section 2, can be generalized to the case when G is a compact Lie group. We discuss the subtle points of this generalization in Section 2.3. In Section 3 we give the definition of quasi-elliptic cohomology $QEll^*_G(-)$ with G a compact Lie group, set up the theory and show its properties. We gave a different definition of $QEll^*_G(-)$ with G a finite group in Definition 3.10, [11], which is equivalent to the definition in this paper. In Section 4, we present the construction of orbifold quasi-elliptic cohomology via the loop space of bibundles. Moreover, we give another construction motivated by Ganter's construction of orbifold Tate K-theory in [7]. The two constructions of orbifold quasi-elliptic cohomology are equivalent.