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Relative entropy in CFT

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Inspired by Edward Witten's questions, we compute the mutual information associated with free fermions, and we deduce many results about entropies for chiral CFT's which are embedded into free fermions, and their extensions. Such relative entropies in CFT are here computed explicitly for the first time in a mathematical rigorous way, and Our results agree with previous computations by physicists based on heuristic arguments; in addition we uncover a surprising connection with the theory of subfactors, in particular by showing that a certain duality, which is argued to be true on physical grounds, is in fact violated if the global dimension of the conformal net is greater than 1.

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1. Introduction

In the last few years there has been an enormous amount of work by physicists concerning entanglement entropies in QFT, motivated by the connections with condensed matter physics, black holes, etc.; see the references in [12] for a partial list of references. However, some very basic mathematical questions remain open. For example, most of the entropies computed in the physics literature are infinite, so the singularity structures, and sometimes the cut off independent quantities, are of most interest. Often, the mutual information is argued to be finite based on heuristic physical arguments, and one can derive the singularities of the entropies from the mutual information by taking singular limits. But it is not even clear that such mutual information, which is well defined as a special case of Araki's relative entropy, is indeed finite.

In this paper we begin to address some of these fundamental mathematical questions motivated by the physicists' work on entropy. For related works, see [12], [19] and [25]. Unlike the main focus in [12], the mutual information considered in our paper can be computed explicitly in many cases and satisfies many conditions, but not all, proposed by physicists such as those in [8]. Our project is strongly motivated by Edward Witten's questions, in particular his question to make physicists' entropy computations rigorous. In this paper we focus on the Chiral CFT in two dimensions, where the results we have obtained are most explicit and have interesting connections to subfactor theory, even though some of our results, such as Theorem 4.4, do not depend on conformal symmetries and apply to more general QFT. The main results are:

1) Theorem 3.18: Exact computation of the mutual information (through the relative entropy as defined by Araki for general states on von Neumann algebras) for free fermions. Note that this was not even known to be finite, for example the main quantity defined in [12] is smaller and does not seem to verify the conditions in the physical literature. Our proof uses Lieb's convexity and the theory of singular integrals; to the best of our knowledge, by Theorem 3.18, Theorem 4.2 and examples in Section 4.4, this is the first time that such relative entropies are computed in a mathematical rigorous way. The results verify earlier computations by physicists based on heuristic arguments, such as P. Calabrese and J. Cardy in [5] and H. Casini and M. Huerta in [9].

In particular, for the free chiral net $\mathcal{A}_r$ associated with r fermions, and two intervals $A = (a_1, b_1)$, $B = (a_2, b_2)$ of the real line, where $b_1 < a_2$, the mutual information associated with A, B is

$$F(A, B) = -\frac{r}{6} \ln \eta \ ,$$

where $\eta = \frac{(b_1-a_2)(b_2-a_1)}{(b_1-a_1)(b_2-a_2)}$ is the cross ratio of A, B, $0 < \eta < 1$.

2) It follows from 1) and the monotonicity of the relative entropy that any chiral CFT in two dimensions that embeds into free fermions, and their finite index extensions, verify most of the conditions (not all, see Section 4.2.1) discussed for example in [8], see Theorem 4.1. This includes a large family of chiral CFTs. Much more can be obtained if