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Quaternionic shearlet transform

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ABSTRACT

The shearlet transform has been shown to be a valuable and powerful time–frequency analyzing tool for optics and non-stationary signal processing. In this article, we propose a novel transform called quaternionic shearlet transform which is designed to represent quaternion-valued signals at different scales, locations and orientations. We investigate the fundamental properties of quaternionic shearlet transform including Parseval's formula, Moyal's principle, inversion formula, and characterization of its range using the machinery of quaternion Fourier transform and quaternion convolution. We conclude our investigation by deriving an analogue of the classical Heisenberg–Pauli–Weyl uncertainty inequality and the associated logarithmic version for the quaternionic shearlet transform.

1. Introduction

Since the inception of wavelets, their importance in the development of science and engineering, particularly in the areas of optical system analysis and non-stationary signal processing, is widely acknowledged (see [1]). Despite of the fact that wavelet transforms have proved to be promising and powerful analyzing tool for one dimensional signals, but the efficiency of the wavelet transforms is considerably reduced when applied to higher dimensional signals as they are not able to efficiently and accurately capture the geometric features like edges and corners at different scales. The detection of such geometric features in signals is often highly desirable in numerous practical applications such as medical imaging, remote sensing, crystallography, and several other areas. To circumvent these limitations, Labate et al. [2] introduced the notion of shearlet transforms in the context of time–frequency and multiscale analysis. Unlike the classical wavelet systems, shearlet systems are non-isotropic in nature, they offer optimally sparse representations, they allow compactly supported analyzing elements, they are associated with fast decomposition algorithms and they provide a unified treatment of continuum and digital data. However, similar to the wavelets, they are an affine-like system of well-localized waveforms at various scales, locations and orientations; that is, they are generated by dilating and translating one single generating function, where the dilation matrix is the product of a parabolic scaling matrix and a shear matrix [3]. Shearlets have been applied in diverse areas of engineering and medical sciences, including inverse problems, computer tomography, image separation and restoration, image deconvolution and thresholding, and medical image analysis [4–8].

On the other hand, considerable attention has been paid for the representation of signals in quaternion domains as quaternion algebra is the closest in its mathematical properties to the familiar system of the real and complex numbers. The quaternion algebra offers a simple and profound representation of signals wherein several components are to be controlled simultaneously. The development of integral transforms for quaternion valued signals has found numerous applications in 3D computer graphics, aerospace

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engineering, artificial intelligence and colour image processing. Due to the non-commutativity of quaternion multiplication, different types of integral transforms have been generalized to quaternion algebra including Fourier and wavelet transforms [9,10], non-harmonic Fourier transform [11], fractional Fourier transform [12], Gabor transform [13], Ridgelet transform [14], Stockwell transform [15] and Curvelet transform [16].

It is well-known that shearlet theory is still in the developing phase and everyday many efforts are being made to extend this theory to a wider class of function spaces. The exciting developments and applications of the shearlet transform along with the profound applicability of the quaternion algebra has inspired us to introduce a new transform namely quaternionic shearlet transform (QSHT) by extending the continuous shearlet transform to the space of quaternion valued functions on $\mathbb{R}^2$. The proposed transform not only inherits the features of shearlet transforms, but also has the capability of signal representations in the quaternion domain. Therefore, the main objective of this article is to introduce the concept of the quaternionic shearlet transform and investigate its different properties using the machinery of quaternion Fourier transforms and quaternion convolution. Moreover, we drive the classical Heisenberg–Pauli–Weyl inequality and logarithmic version of this inequality for the quaternionic shearlet transforms. It is hoped that this transform might be useful in three dimensional color field processing, space color video processing, crystallography, aerospace engineering, oil exploration and for the solution of many types of quaternionic differential equations.

The article is organized as follows: We begin in Section 2 by presenting the notation, quaternion algebra and shearlet theory needed to understand and place our results in context. In Section 3, we introduce the concept of quaternionic shearlet transform and obtain the expected properties of the extended shearlet transform including Parseval's formula, Moyal's principle, inversion formula, and characterization of its range. The well known Heisenberg–Pauli–Weyl inequality and logarithmic uncertainty principle are generalized in the quaternion Fourier domains in Section 4. Finally conclusions are summarized in Section 5.

2. Quaternion algebra and shearlet transform

The theory of quaternions was initiated by the Irish mathematician Sir W.R. Hamilton in 1843 and is denoted by $\mathbb{H}$ in his honour. The quaternion algebra provides an extension of the complex number system to an associative non-commutative four-dimensional algebra. The quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ is given by

$$\mathbb{H} = \Big\{\mathbf{h} = a_0 + i\,a_1 + j\,a_2 + k\,a_3 : \; a_0,\, a_1,\, a_2,\, a_3 \in \mathbb{R}\Big\},$$

where i, j, k denote the three imaginary units, obeying the Hamilton's multiplication rules

$$\mathrm{ij} = k = -\mathrm{ji},\; \mathrm{jk} = i = -\mathrm{kj},\; \mathrm{ki} = j = -\mathrm{ik},\; \text{and}\; i^2 = j^2 = k^2 = \mathrm{ijk} = -1.$$

For quaternions $\mathbf{h}_1 = a_0 + i\,a_1 + j\,a_2 + k\,a_3$ and $\mathbf{h}_2 = b_0 + i\,b_1 + j\,b_2 + k\,b_3$, the addition is defined component-wise and the multiplication is defined as

$$\begin{aligned}\mathbf{h}_1\mathbf{h}_2 &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + i\,(a_1b_0 + a_0b_1 + a_2b_3 - a_3b_2)\\ &\quad + j\,(a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3) + k\,(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1).\end{aligned}$$

The conjugate and norm of a quaternion $\mathbf{h} = a_0 + i\,a_1 + j\,a_2 + k\,a_3$, are given by $\overline{\mathbf{h}} = a_0 - i\,a_1 - j\,a_2 - k\,a_3$ and $\|\mathbf{h}\|_{\mathbb{H}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$, respectively. We also note that an arbitrary quaternion $\mathbf{h}$ can be represented by two complex numbers as $\mathbf{h} = (a_0 + i\,a_1) + j\,(a_2 - i\,a_3) = u + j\,v$, where $u, v \in \mathbb{C}$, and hence, $\overline{\mathbf{h}} = \bar{u} - j\,v$, with $\bar{u}$ denoting the complex conjugate of u. Moreover, the inner product of any two quaternions $\mathbf{h}_1 = u_1 + jv_1$, and $\mathbf{h}_2 = u_2 + j\,v_2$ in $\mathbb{H}$ is defined by

$$\langle \mathbf{h}_1, \mathbf{h}_2\rangle_{\mathbb{H}} = \mathbf{h}_1\overline{\mathbf{h}_2} = (u_1\bar{u}_2 + \bar{v}_1v_2) + j\,(v_1\bar{u}_2 - \bar{u}_1v_2).$$

By virtue of the complex domain representation, a quaternion-valued function $F\colon \mathbb{R}^2 \to \mathbb{H}$ can be decomposed as $F(x) = f_1 + jf_2$, where f_1, f_2 are both complex valued functions. Here it is appropriate to point out that the following notations will be followed in the rest of the article, $\check{F}(x) = F(-x)$ and $\widetilde{F}(x) = \overline{f_1}(x) - j\check{f_2}(x)$.

Let us denote $L^2(\mathbb{R}^2, \mathbb{H})$, the space of all quaternion valued functions F satisfying

$$\|F\|_2 = \left\{\int_{\mathbb{R}^2}\Big(|f_1(x)|^2 + |f_2(x)|^2\Big)\,\mathrm{dx}\right\}^{1/2} < \infty\,.$$

The norm on $L^2(\mathbb{R}^2, \mathbb{H})$ is obtained from the inner product of the quaternion valued functions $F = f_1 + jf_2$, and $G = g_1 + j\,g_2$ as

$$\begin{aligned}\langle F, G\rangle_2 &= \int_{\mathbb{R}^2}\langle F, G\rangle_{\mathbb{H}}\,\mathrm{dx}\\ &= \int_{\mathbb{R}^2}\Big\{\Big(f_1(x)\overline{g_1}(x) + \overline{f_2}(x)g_2(x)\Big) + j\Big(f_2(x)\overline{g_1}(x) - \overline{f_1}(x)g_2(x)\Big)\Big\}\,\mathrm{dx}.\end{aligned}$$

An easy computation shows that $L^2(\mathbb{R}^2, \mathbb{H})$ equipped with above defined inner product is a Hilbert space.

Definition 2.1. For any quaternion valued function $F \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$, the quaternion Fourier transform (QFT) is denoted by $\mathcal{F}_q$ and is given by

$$\mathcal{F}_q[F(t)](\xi) = \hat{F}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi\mathbf{i}x_1\cdot\xi_1}F(x)\,e^{-2\pi\mathbf{j}x_2\cdot\xi_2}\,\mathrm{dx}, \tag{2.1}$$

where $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$ and the quaternion exponential product $e^{-2\pi\mathbf{i}x_1\cdot\xi_1}e^{-2\pi\mathbf{j}x_2\cdot\xi_2}$ is the quaternion Fourier kernel. The