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# Exponential probability distribution on symmetric matrices

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ABSTRACT

In this paper, we define in the most natural way a multivariate extension of the exponential distribution as a particular Wishart on the cone of positive definite symmetric matrices. We also introduce a notion of reliability function for a matrix random variable. We then show that, under a condition of invariance, the exponential distribution on symmetric matrices is characterized by a property of memoryless generalizing and including the characterization established for the real exponential distribution.



## 1. Introduction

The exponential probability distribution on the real line is one of the most common probability distributions. Although it is a particular gamma, it has many important specific probabilistic characterizations, the most celebrated are the characterization by memoryless and the characterization by independence between the fractional part and integer part. The exponential distribution has also a wide range of statistical applications in many fields. The monograph of Balakrishnan (1996) provides a review of the literature on this distribution and on its applications. In the multivariate setting, there is not a unique way to define a multivariate exponential distribution, in fact, different statistical requirements have led to different derivations of this distribution. For a survey and a discussion on the most important multivariate extensions of the exponential distribution and their applications, we refer the reader to Basu (1988). It is well known that the absolutely continuous probability distributions concentrated on the positive half real line usually extend to distributions on the cone of positive definite symmetric matrices or more generally on any symmetric cone. For instance, the real gamma extends to the Wishart or to the Riesz distribution (see Hassairi and Lajmi, 2001). The same thing happens with the beta, the Dirichlet, the inverse Gaussian and many other distributions (Gupta and Nagar, 1999). In analogy with the real case where the exponential distribution appears as a gamma distribution with a particular shape parameter, we define in the present paper the exponential distribution on the cone of positive definite symmetric matrices as a particular Wishart. We think that this is the most natural way to define a multivariate extension of the exponential distribution. We also define a notion of reliability function for a random variable on the cone of positive definite symmetric matrices, and we use it to show that the exponential distribution has the property of memoryless. Using some results from harmonic analysis on symmetric cones in particular a notion of Mellin transform deduced from the spherical Fourier transform, we show that this property of memoryless characterizes the exponential distribution on symmetric matrices. It is worth mentioning here that the so-called matrix exponential distribution in the literature is not in fact concentrated on a matrix space, but it is an exponential probability distribution on the real line such that the intensity parameter is a matrix. A discussion on this distribution is given in Chapter 3 of Lipsky (1992), one may also refer to Bladt and Nielsen (2017).

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Throughout, we will denote by $V$ the space of symmetric matrices of order $r$, by $n$ its dimension. We denote the identity matrix by $e$, the determinant of an element $x$ of $V$ by $\Delta(x)$ and its trace by $\mathrm{tr}x$, and we equip $V$ with the scalar product

$$\langle x, y\rangle = \mathrm{tr}(xy).$$

Let $\Omega$ be the cone of positive definite elements of $V$. Then we will write $x > 0$, when $x \in \Omega$ and $x < 0$, when $x \in -\Omega$.

For $x$ and $a$ in $\Omega$, we write $x > a$ when $x - a > 0$, in other words $x \in (\Omega + a)$. Similarly, $x < a$ means that $x \in (a - \Omega)$.

Now, for an invertible $r \times r$ matrix $a$ we consider the automorphism $g_a$ of $V$ defined by

$$g_a(x) = axa^*,$$

where $a^*$ is the transpose of $a$.

We denote by $G$ the group of such isomorphisms and by $K$ the subgroup of elements of $G$ corresponding to $a$ orthogonal, called the orthogonal group. Also the set of elements $g_a$ of $G$ such that $a$ is lower triangular is a subgroup of $G$ called the triangular group and denoted by $T$.

To define a notion of ratio in $\Omega$, we need a division algorithm which is a map $g : \Omega \to G$ such that $g_x(x) = e$, for all $x$ in $\Omega$. The most common division algorithms are the one using the quadratic representation and the one using the Cholesky decomposition.

The quadratic representation consists in writing in a unique manner an element $y$ in $\Omega$ as $y = y^{1/2}y^{1/2}$ and then consider the map $y \to P(y^{\frac{1}{2}})$, where $P(y^{1/2})x = y^{1/2}xy^{1/2}$. This defines a multiplication algorithm and the map $y \to P(y^{-1/2})$ where $P(y^{-1/2})x = y^{-1/2}xy^{-1/2}$ defines a division algorithm.

The other division algorithm uses the Cholesky decomposition of an element $y$ of $\Omega$, that is on the fact $y$ can be written in a unique manner as $y = tt^*$, where $t$ is a lower triangular matrix with a strictly positive diagonal.

For an element $x$ in $V$, we set $\pi(y)x = txt^*$, and we define the "quotient" of $x$ by $y$ as

$$\pi^{-1}(y)x = t^{-1}xt^{*-1}.$$

We will also use the notation

$$\pi^*(y)x = t^*xt \ \text{ and } \ \pi^{*-1}(y)x = t^{*-1}xt^{-1}.$$

Now, for $x = (x_{ij})_{1\le i,j\le r}$ in $\Omega$ and $1 \le k \le r$, let $\Delta_k(x)$ denote the principal minor of order $k$ of $x$, that is the determinant of the sub-matrix $(x_{ij})_{1\le i,j\le k}$. Then the generalized power of $x$ is defined in Faraut and Korànyi (1994), page 122, for $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ and $x \in \Omega$ by

$$\Delta_s(x) = \Delta_1^{s_1-s_2}(x)\Delta_2^{s_2-s_3}(x)\ldots\Delta_r^{s_r}(x).$$

When the vector $s$ is such that $s_1 = s_2 = \cdots = s_r = p$, we have

$$\Delta_s(x) = \Delta^p(x).$$

The generalized power function has the following useful property (see Massam, 1994 and Hassairi et al., 2008)

$$\Delta_s(\pi(x)y) = \Delta_s(x)\Delta_s(y). \tag{1.1}$$

## 2. Reliability function

In this section, we introduce a notion of reliability function for a random variable $X$ in the cone $\Omega$ of positive definite symmetric matrices and we show that under a condition of invariance this function characterizes the distribution of $X$.

**Definition 2.1.** Let $X$ be a random variable $\Omega$. Then the function

$$\overline{F}_X(x) = P(X > x) = P(X \in x + \Omega) \tag{2.2}$$

is called the reliability function of $X$.

**Definition 2.2.** We say that a random variable $X$ in $\Omega$ is invariant by the orthogonal group $K$ if, for all $k \in K$, the random variables $X$ and $k(X)$ have the same distribution. For $\alpha \in \Omega$, we say that $X$ is $(\alpha, K)$-invariant if $\pi^*(\alpha)X$ is $K$-invariant.

**Definition 2.3.** The Mellin transform of a random variable $X$ in $\Omega$ or of the distribution of $X$ is the map defined for $s = (s_1, s_2, \ldots, s_r)$ in $\mathbb{R}^r$ by

$$M_X(s) = E(\Delta_s(X)). \tag{2.3}$$

The Mellin transform defined in this way for a random variable in $\Omega$ is in fact related to the notion of spherical Fourier transform in a symmetric cone. It is shown (see Hassairi and Regaig, 2009) that if $X$ is $K$-invariant and if the map $s \mapsto M_X(s)$ is defined on an open subset of $\mathbb{R}^r$, then it characterizes the distribution of $X$.



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