Statistics and Probability Letters 145 (2019) 74–80

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Linear process bootstrap unit root test

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ARTICLE INFO

Article history:
Received 7 January 2018
Received in revised form 3 August 2018
Accepted 14 August 2018
Available online 7 September 2018

Keywords:
Integrated time series
Size distortion
Resampling
Functional central limit theorem

ABSTRACT

One of the most widely applied unit root tests suffers from size distortions when moving average noise exists. As a remedy, this paper proposes a bootstrap test targeting moving average noise and shows its effectiveness in both theory and simulation.

1. Introduction

Throughout extensive literature concerning unit root test, the Augmented Dickey–Fuller (ADF) test and the Phillips–Perron (PP) test are perhaps the most renowned. When put into simulation, the PP test enjoys higher power than the ADF test but suffers greater size distortion, especially under negative Moving Average (MA) noise (Phillips and Perron, 1988; Nabeya and Perron, 1994; Cheung and Lai, 1997; Leybourne and Newbold, 1999). For a solution to this size distortion occurrence, see Perron and Ng (1996).

Here we propose a bootstrap unit root test as a remedy. When the asymptotic distributions of the test statistics involve unknown parameters, bootstrap circumvents the estimation of the unknown parameters and thus facilitates hypothesis testing. On the other hand, when the asymptotic distributions are pivotal, a bootstrap unit root test may enjoy second order efficiency, and may consequently reduce the aforementioned size distortion (Park, 2003). Variants of the bootstrap unit root test include the AutoRegressive (AR) sieve bootstrap test (Psaradakis, 2001; Chang and Park, 2003; Paparoditis and Politis, 2005; Palm et al., 2008), the block bootstrap test (Paparoditis and Politis, 2003; Parker et al., 2015), the stationary bootstrap test (Swensen, 2003; Parker et al., 2006), and the wild bootstrap test (Cavaliere and Taylor, 2009).

To target the size distortion of the PP test under MA noise, we apply the Linear Process Bootstrap (LPB) of McMurry and Politis (2010) to the unit root test. As the closest analogue to the MA-sieve bootstrap, LPB first estimates the autocovariance matrix by fitting a MA-type autocovariance function, then pre-whitens the noise with the estimated autocovariance matrix, then bootstraps from the pre-whitened noise, and finally post-colors the bootstrap noise with the estimated autocovariance matrix. In the sample mean case, McMurry and Politis (2010); Jentsch et al. (2015) indicate good asymptotic and empirical performance of LPB, particularly in the presence of MA noise.

As a result, the LPB unit root test becomes a promising solution to the size distortion under MA noise. We proceed to develop a large sample theory for the LPB unit root test by establishing a bootstrap Functional CLT (FCLT) for LPB. Despite its name, the LPB unit root test turns out to be asymptotically valid under not only linear noises but also a large family of non-linear noises, namely, the physical dependent process defined in Wu (2005).

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https://doi.org/10.1016/j.spl.2018.08.006
0167-7152/

This paper proceeds as follows: Section 2 specifies the physical dependence assumption and recalls the Phillips–Perron test. Section 3 introduces the LPB unit root test, details the estimation of the autocovariance matrix, and describes the adaptive bandwidth selection. Section 4 presents the empirical results of the LPB unit root test. Supplementary materials include all technical proofs, as well as a discussion concerning deterministic components.

2. Phillips–Perron test

Suppose $\{Y_t\}_{t=1}^n$ is an observable sequence of random variables. For $t = 1, 2, \ldots,$ define ϕ_t and V_t as the prediction coefficient and the prediction error, respectively, when predicting Y_t with Y_{t-1}. Now suppose $\phi_t = \phi$ for all $t = 1, 2, \ldots.$ Then

$$Y_t = \phi Y_{t-1} + V_t. \tag{2.1}$$

Denote the set of integers by $\mathbb{Z}$. We now assume that the noise sequence $\{V_t\}_{t\in\mathbb{Z}}$ is strictly stationary, short-range dependent, and invertible. Specifically, consider the following assumptions on $\{V_t\}_{t\in\mathbb{Z}}$:

Assumption 2.1. Let $\{\epsilon_t\}_{t\in\mathbb{Z}}$ be a sequence of independent and identically distributed (iid) random variables. Let ϵ_0' be identically distributed with ϵ_0, and be independent of $\{\epsilon_t\}_{t\in\mathbb{Z}}$. Suppose $V_t = g(\ldots, \epsilon_{t-1}, \epsilon_t)$. Let $V_t' = g(\ldots, \epsilon_{-1}, \epsilon_0', \epsilon_1, \ldots, \epsilon_t)$, $\delta_p(t) = (E(|V_t - V_t'|^p))^{1/p}$ be the physical dependence measure of $\{V_t\}$, and $\gamma(h) = E(V_t V_{t-h})$. Suppose $\sum_{h\in\mathbb{Z}}\gamma(h) > 0$, $\sum_{h=0}^{\infty} h|\gamma(h)| < \infty$, $E(V_t) = 0$, $E(V_t^4) < \infty$, and $\sum_{t=1}^{\infty}\delta_4(t) < \infty$.

Assumption 2.2. Recall Assumption 2.1. Further, assume that for some $p > 4$, $\sum_{t=1}^{\infty}\delta_p(t) < \infty$ and $E(|V_t|^p) < \infty$; for some $\beta > 2$, $|\gamma(h)| = o(h^{-\beta})$; and for some $\alpha > 0$, $h^\alpha \sum_{k=h+1}^{\infty}|\gamma(k)|$ is non-increasing when h is large enough.

When $\phi = 1$, suppose $Y_0 = 0$. Then $\{Y_t\}_{t=1}^{\infty}$ is a unit root process starting at zero. When $\phi < 0$, suppose (2.1) holds for all $t \in \mathbb{Z}$. Then $\{Y_t\}_{t=1}^{\infty}$ is a strictly stationary process. To separate these two cases, we test $H_0 : \phi = 1$ vs $H_1 : \phi < 1$. Let $\hat{\phi}$ be the Ordinary Least Squares (OLS) estimator in $Y_t = \hat{\phi}Y_{t-1} + \hat{V}_t$, and $t_{\hat{\phi}}$ be the t-statistic of $\hat{\phi}$. Let $\sigma^2 = \lim_{n\to\infty} Var(n^{-1/2}\sum_{t=1}^n V_t)$, $\hat{\sigma}^2$ be a consistent estimator of σ^2, $\hat{\gamma}(0)$ be a consistent estimator of $\gamma(0)$, and

$$\begin{aligned} Z_\phi &= n(\hat{\phi} - 1) - (\hat{\sigma}^2 - \hat{\gamma}(0))(2n^{-2}\sum_{t=1}^n Y_{t-1}^2)^{-1}, \\ Z_t &= (\hat{\gamma}(0)/\hat{\sigma}^2)t_{\hat{\phi}} - (1/2)(\hat{\sigma}^2 - \hat{\gamma}(0))(\hat{\sigma}^2 n^{-2}\sum_{t=1}^n Y_{t-1}^2)^{-1/2}. \end{aligned} \tag{2.2}$$

Let $W(u)$ be a standard Brownian motion. By Theorem 3 of Wu (2005), under the null hypothesis H_0 and Assumption 2.1 we have

$$\begin{aligned} Z_\phi &\Rightarrow (\int_0^1 W(u)dW(u))(\int_0^1 (W(u))^2 du)^{-1}, \\ Z_t &\Rightarrow (\int_0^1 W(u)dW(u))(\int_0^1 (W(u))^2 du)^{-1/2}. \end{aligned} \tag{2.3}$$

The PP test rejects the null hypothesis H_0 when Z_ϕ is too small, or, alternatively, when Z_t is too small, and calculates the critical values from (2.3).

3. Linear process bootstrap unit root test

As mentioned in the introduction, the PP test enjoys high empirical power, but suffers from empirical size distortions under negative MA noise. To mitigate the size distortion while preserving the high power, we apply LPB to the OLS estimator $\hat{\phi}$ and its t-statistic $t_{\hat{\phi}}$. Alternatively, we can also apply LPB to Z_ϕ and Z_t in (2.2); however, in Section 4, applying LPB to Z_ϕ or Z_t gives an inferior empirical result.

Let $\bar{Y} = n^{-1}\sum_{t=1}^n Y_t$, $\bar{V} = n^{-1}\sum_{t=1}^n V_t$, $\bar{\hat{V}} = n^{-1}\sum_{t=1}^n \hat{V}_t$, $\bar{\hat{\epsilon}} = n^{-1}\sum_{t=1}^n \hat{\epsilon}_t$, and $\hat{\sigma}_{\hat{\epsilon}}^2 = n^{-1}\sum_{t=1}^n(\hat{\epsilon}_t - \bar{\hat{\epsilon}})^2$. Let $\boldsymbol{V} = (V_1, \ldots, V_n)'$, $\check{\boldsymbol{V}} = (\check{V}_1, \ldots, \check{V}_n)'$, $\hat{\boldsymbol{V}} = (\hat{V}_1, \ldots, \hat{V}_n)'$, $\bar{\hat{\boldsymbol{V}}} = (\bar{\hat{V}}, \ldots, \bar{\hat{V}})'$, $\boldsymbol{V}^* = (V_1^*, \ldots, V_n^*)'$, $\hat{\boldsymbol{\epsilon}} = (\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n)'$, and $\boldsymbol{\epsilon}^* = (\epsilon_1^*, \ldots, \epsilon_n^*)'$. Let $\boldsymbol{\Sigma} = Var(\boldsymbol{V})$ and $\hat{\boldsymbol{\Sigma}}_{\hat{V}}$ be a positive definite estimator of $\boldsymbol{\Sigma}$. We will further specify $\hat{\boldsymbol{\Sigma}}_{\hat{V}}$ in Algorithm 3.2. Let $\hat{\boldsymbol{\Sigma}}_{\hat{V}}^{1/2}$ be a lower triangular matrix that satisfies Cholesky decomposition $\hat{\boldsymbol{\Sigma}}_{\hat{V}}^{1/2}\hat{\boldsymbol{\Sigma}}_{\hat{V}}^{1/2'} = \hat{\boldsymbol{\Sigma}}_{\hat{V}}$, and $\hat{\boldsymbol{\Sigma}}_{\hat{V}}^{-1/2}$ be the inverse matrix of $\hat{\boldsymbol{\Sigma}}_{\hat{V}}^{1/2}$. Let P^*, E^*, Var^*, Cov^* be the probability, expectation, variance, and covariance, respectively, conditional on data $\{Y_t\}$.

Algorithm 3.1 (*Linear Process Bootstrap Unit Root Test*).

Step 1: regress $Y_t = \hat{\phi}Y_{t-1} + \hat{V}_t$; record $\hat{\phi}$ and its t-statistic $t_{\hat{\phi}}$.
Step 2: let $\check{V}_t = \hat{V}_t - \bar{\hat{V}}$, $\hat{\boldsymbol{\epsilon}} = \hat{\boldsymbol{\Sigma}}_{\hat{V}}^{-1/2}\check{\boldsymbol{V}}$, and $\check{\epsilon}_t = (\hat{\epsilon}_t - \bar{\hat{\epsilon}})/\hat{\sigma}_{\hat{\epsilon}}$.
Step 3: randomly sample $\epsilon_1^*, \ldots, \epsilon_n^*$ from $\{\check{\epsilon}_1, \ldots, \check{\epsilon}_n\}$.