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Higher orders of Magnus expansion for point kinetics telegraph model

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ABSTRACT

Neutron transport has been still under very active development in research institutions and academia throughout the world. The spatial, temporal, energy and directional angle dependence make it remains one of the most computationally challenging problems in the world. The correct P_1 approximation to the neutron transport equation is not the first order diffusion equation, but the second-order in time, which is called telegraph equation. In this paper, a synopsis derivation of the point kinetics telegraph model is obtained from the neutron transport equation as a couple system of stiff differential equations. The problem of the obtained system is formulated in the matrix form and solved by higher orders Magnus expansion, where the first successive three orders of Magnus expansion are derived analytically for the point telegraph equations with multi-group of delayed neutrons and various different types of reactivity. These approximations depend on the exponential function of Magnus matrix, where the calculations are obtained using the eigenvalues of Magnus matrix and the corresponding eigenvectors. The proposed methods are tested using different cases of reactivity such as step, ramp and sinusoidal reactivity insertions. The numerical results indicate that the high order of Magnus expansion approximations is accurate compared with the traditional methods.

1. Introduction

The determination of the distribution of neutrons is the central problem of nuclear reactor theory. In fact, the physical operations in a nuclear reactor permanently depend on the distribution (Duderstadt and Hamilton, 1976) of neutron density, whose mathematical description is based on the balance equation i.e. neutron-transport equation (Hetrick, 1971; Stacey, 2007). The mathematical category of this equation is integro-differential one, and the required statistical distribution of neutrons flux depends on the variables: time; energy; spatial and angular direction. As a rule, to get a satisfactory solution, the simplified forms are used for practical calculations of the neutron transport equation of nuclear reactors. Most reactor studies developed the neutron behavior as a diffusion process. The modified equation that is known as a multigroup diffusion methodology (Lewis and Miller, 1993; Marchuk, G.I. and Lebedev, V.I., 1986; Cho, 2005; Sutton, T.M. and Aviles, B.N., 1996) are generally used for reactor analysis, which is also utilized in other branches of engineering calculation codes. Further, many of the most useful neutron models obtained from approximations to the neutron transport equation reduce to familiar form, such as telegraph equations. These simplifications result from the elimination of various independent variables in the general formulation are usually introduced by virtue of certain limitations imposed on the physical situation. The feature of this equation is that it describes physical phenomena which exhibit both wavelike characteristics and residual disturbance effects (Robert and David, 1960). The wave like behavior is given by the second-order term in t, and the residual disturbance effect, by the first-order term in t. Wave effects propagate with a finite velocity, and time-dependent neutron phenomena should properly include such a term since time-wise variations in neutron population cannot be sensed at a given distance from the source in less time than it takes a neutron of speed v to reach that point. However, after the passage of this "wave," the disturbance persists and this effect is described by the first-order term. Moreover, The system of the neutron telegraph equations represents the modified system of the point kinetics equations which is one of the most important systems in the nuclear engineering derived from the neutron transport equation (Weinberg and Noderer, 1951; Weinberg and wigner, 1958; Meghreblian and Holmes, 1960; Beckurts and Wirtz, 1964). Recently, the successive publications (Altahhan et al., 2016; 2017a,b,c) developed a telegraph model of the point reactor kinetics (TPRK) based on the mono-energetic, non-fractional order telegraph approximation of the neutron transport equation. The modified model was solved for several cases of time varying reactivities and temperature feedback

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(non-linear system model), while comparing it to that of the diffusion point reactor kinetics model in an infinite thermal homogenous nuclear reactor, (Altahhan et al, 2017a). Furthermore, the proposed system of non-linear telegraph model was solved analytically for a linear insertion of reactivity, which typically introduced by lifting the control rods discontinuously and manually during the cold start-up of a subcritical nuclear reactor. In fact, the telegraph model introduces a new parameter called the relaxation time (s). The analysis was extended to study its impact on the analytical solution for this case of reactivity insertion and for several speeds of lifting the control rods. The modified telegraph system proved that as the relaxation time increases, the solution response is relaxed behind that of the diffusion model (Altahhan et al., 2017b).

In this paper, the derived neutron point telegraph kinetics model is developed for the finite nuclear reactor system. A mathematical approach based on the successive approximations of Magnus expansion was developed and formulated to obtain a complete solution of the telegraph point reactor kinetics model (TPRK). The model was applied and solved for the case in which a step insertion of reactivity is introduced into a finite and infinite thermal nuclear reactor as well as developed a matrix form of the model. The paper is organized as follows: the point kinetics telegraph model was derived from the neutron transport equation in section 2. The developed stiff coupled differential equations were formulated in the matrix form of the differential equations in section 3. Section 4 includes the solution of the presented system based on the approximations of Magnus expansion for the different case of time-varying reactivities, where section 5 comprises of the analysis of results and discussions. Finally, the conclusion is given in section 6.

2. The point telegraph kinetics system

The determination of the distribution of neutrons in the nuclear reactor that determines the rate at which various nuclear reaction occurs within the reactor generally depends on the variables: position, energy, direction and time. The neutron transport equation along with its initial condition in terms of the angular flux is (Duderstadt and Hamilton, 1976)

$$\frac{1}{v}\frac{\partial\varphi(\mathbf{r},E,\hat{\Omega},t)}{\partial t}+\hat{\Omega}.\nabla\varphi(\mathbf{r},E,\hat{\Omega},t)+\Sigma_t\varphi(\mathbf{r},E,\hat{\Omega},t)$$
$$=\int_{4\pi}\int_0^\infty\Sigma_s\left(E'\to E,\widehat{\Omega'}\to\hat{\Omega}\right)\varphi\left(\mathbf{r},E',\widehat{\Omega'},t\right)dE'd\widehat{\Omega'}+s(\mathbf{r},E,\hat{\Omega},t) \tag{1}$$

where the initial condition is

$$\varphi(\mathbf{r},E,\hat{\Omega},0)=\varphi_0(\mathbf{r},E,\hat{\Omega}), \tag{2}$$

where, $\varphi(\mathbf{r},E,\hat{\Omega},t)$ is the angular flux as a function of the position vector $\mathbf{r}$, time t, the neutron direction of motion $\hat{\Omega}$ and energy E, v is the neutron speed, Σ_t is the total cross section, $\Sigma_s\left(E'\to E,\widehat{\Omega'}\to\hat{\Omega}\right)$ is scattering cross section from energy E' to E and from direction Ω' to Ω, $s(\mathbf{r},E,\hat{\Omega},t)$ is the neutron source term and r_s denotes a point on the surface.

The neutron transport equation has seven independent variables $(\mathbf{r},E,\hat{\Omega},t)$ and any attempt to solve the transport equation for a realistic system without an approximation is incompetent and need a heavy use of a digital computer. Assuming that the neutron energy doesn't change in a scattering collision, the scattering cross section takes the form

$$\Sigma_s\left(E'\to E,\widehat{\Omega'}\to\hat{\Omega}\right)=\Sigma_s\left(E,\widehat{\Omega'}\to\hat{\Omega}\right)\delta(E'-E), \tag{3}$$

where $\delta(E'-E)$ is the Dirac δ-function defined by the property

$$\int f(x')\delta(x-x')dx'=f(x). \tag{4}$$

That is means, the scattering term in the transport equation is simplified to:

$$\int_{4\pi}\int_0^\infty\Sigma_s\left(E'\to E,\widehat{\Omega'}\to\hat{\Omega}\right)\varphi\left(\mathbf{r},E',\widehat{\Omega'},t\right)dE'd\widehat{\Omega'}$$
$$=\int_{4\pi}\Sigma_s\left(E,\widehat{\Omega'}\to\hat{\Omega}\right)\varphi\left(\mathbf{r},E,\widehat{\Omega'},t\right)d\widehat{\Omega'}=\Sigma_s(E,\hat{\Omega})\varphi(\mathbf{r},E,\hat{\Omega},t). \tag{5}$$

When certain assumptions are made for neutron transport equation about the energy dependence of the neutron flux, i. e by assuming that one can portray the neutrons by a single energy or speed, the explicit dependence on energy is eliminated to write the one speed neutron transport equation as:

$$\frac{1}{v}\frac{\partial\varphi(\mathbf{r},\hat{\Omega},t)}{\partial t}+\hat{\Omega}.\nabla\varphi(\mathbf{r},\hat{\Omega},t)+\Sigma_t\varphi(\mathbf{r},\hat{\Omega},t)=\Sigma_s(\hat{\Omega})\varphi(\mathbf{r},\hat{\Omega},t)+s(\mathbf{r},\hat{\Omega},t). \tag{6}$$

If the integral on both sides is regarded over $\hat{\Omega}$, the explicit form taken by the neutron conservation equation will be (Duderstadt and Hamilton, 1976):

$$\frac{1}{v}\frac{\partial}{\partial t}\Phi(\mathbf{r},t)+\nabla.\mathbf{J}(\mathbf{r},t)+\Sigma_t\Phi(\mathbf{r},t)=\Sigma_s\Phi(\mathbf{r},t)+S(\mathbf{r},t), \tag{7}$$

where, $\Phi(\mathbf{r},t)\equiv\int_{4\pi}\varphi(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}$ is the zero moment of the angular flux, which is the scalar flux. $\mathbf{J}(\mathbf{r},t)\equiv\int_{4\pi}\hat{\Omega}\varphi(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}$ is the first moment of the angular flux, which is the neutron current density.

After mathematical manipulated of the one-speed neutron transport equation by multiplying equation (6) by $\hat{\Omega}$ followed by integrating over the direction, the corresponding equation for the current density $\mathbf{J}(\mathbf{r},\mathbf{t})$ in the one-speed case is obtained as:

$$\frac{1}{v}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)+\nabla.\int_{4\pi}\hat{\Omega}\hat{\Omega}\varphi(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}+\Sigma_t\mathbf{J}(\mathbf{r},t)=\bar{\mu}_0\Sigma_s\mathbf{J}(\mathbf{r},t)+S_1(\mathbf{r},t), \tag{8}$$

where, $\bar{\mu}_0=\left\langle\hat{\Omega}.\widehat{\Omega'}\right\rangle$ is the average scattering angle cosine, and $S_1(\mathbf{r},t)\equiv\int_{4\pi}\hat{\Omega}s(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}$ is the first moment of the neutron source.

Let us assume the following successive assumption. First, the angular flux is only linearly anisotropic i.e. depends weakly only on the angle which means that $\nabla.\int_{4\pi}\hat{\Omega}\hat{\Omega}\varphi(\mathbf{r},\hat{\Omega},t)d\hat{\Omega}=\frac{1}{3}\nabla\Phi(\mathbf{r},t)$. Second, the neutron source is isotropic, so that $S_1(\mathbf{r},t)=0$. Third, the nuclear reactor is homogenous which means that the total and scattering cross section are independent on the space. The effect of moving the control rod on the absorption cross section $\Sigma_a=\Sigma_t-\Sigma_s$ depend only on time. Considering the previous assumptions, equations (7) and (8) take the following form:

$$\frac{1}{v}\frac{\partial\Phi(\mathbf{r},t)}{\partial t}+\nabla\cdot\mathbf{J}(\mathbf{r},t)+\Sigma_a\Phi(\mathbf{r},t)=S(\mathbf{r},t) \tag{9}$$

$$\frac{1}{v}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)+\Sigma_{tr}\mathbf{J}(\mathbf{r},t)+\frac{1}{3}\nabla\Phi(\mathbf{r},t)=0, \tag{10}$$

where $\Sigma_{tr}=\Sigma_t-\bar{\mu}_0\Sigma_s$ is the transport cross section.

Equations (9) and (10) are known in nuclear reactor analysis as the P_1 approximation (in the one-speed approximation). The neutron diffusion equation is based on the Fick's law $\mathbf{J}(\mathbf{r},t)=-D\nabla\Phi(\mathbf{r},t)$ (Robert and David, 1960), which implies that the current density $\mathbf{J}(\mathbf{r},t)$ adjusts instantaneously to the gradient of the scalar flux $\Phi(\mathbf{r},t)$. By the combination of a continuity equation and a constitutive equation (Fick's first law) one arrives at the diffusion equation (or second Fick's law). The solution of this constitutive equation shows that even for very small times, there exists a finite amount of the diffusing substance at large distances from the origin (Albert and Metzlerz, 1997), that it issues an