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Super vertex algebras, meromorphic Jacobi forms and umbral moonshine

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ABSTRACT

The vector-valued mock modular forms of umbral moonshine may be repackaged into meromorphic Jacobi forms of weight one. In this work we constructively solve two cases of the meromorphic module problem for umbral moonshine. Specifically, for the type A Niemeier root systems with Coxeter numbers seven and thirteen, we construct corresponding bigraded super vertex operator algebras, equip them with actions of the corresponding umbral groups, and verify that the resulting trace functions on canonically twisted modules recover the meromorphic Jacobi forms that are specified by umbral moonshine. We also obtain partial solutions to the meromorphic module problem for the type A Niemeier root systems with Coxeter numbers four and five, by constructing super vertex operator algebras that recover the meromorphic Jacobi forms attached to maximal subgroups of the corresponding umbral groups.

Contents

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1. Introduction

Eguchi–Ooguri–Tachikawa initiated a new phase in moonshine with their observation [18] that representations of the largest sporadic Mathieu group M_{24} are visible in the multiplicities of irreducible superconformal algebra modules in the K3 elliptic genus. The generating function of these multiplicities is a mock modular form $H^{(2)}$ of weight $\frac{1}{2}$ (cf. [15]). Once *twined* counterparts $H_g^{(2)}$ for $g \in M_{24}$ had been identified [8,24,25,17] and characterized [3], Gannon was able to confirm [23] that there is a corresponding graded M_{24}-module, for which the q-series of $H^{(2)} = H_e^{(2)}$ is the graded dimension. But so far there has been no explicit construction of this *Mathieu moonshine* module, such as might be compared to the vertex operator algebra of monstrous moonshine [10,30,31] that was discovered by Frenkel–Lepowsky–Meurman [20–22], and used to prove the monstrous moonshine conjectures by Borcherds [2]. The purpose of this paper is to solve a closely related construction problem, for some closely related instances of moonshine.

To motivate our approach we recall the curious circumstance that the *McKay–Thompson series* $H_g^{(2)}$ of Mathieu moonshine may be repackaged into modular forms of different kinds. Indeed, if $\chi_g^{(2)}$ is the number of fixed points of $g \in M_{24}$ in the defining permutation representation on 24 points, then

$$Z_g^{(2)}(\tau, z) := \chi_g^{(2)} \frac{\mu_{2,0}(\tau, z)}{\mu_{1,0}(\tau, z)} + H_g^{(2)}(\tau) \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \tag{1.1}$$

is a weak Jacobi form of weight 0, index 1, and some level depending on g (where $\mu_{m,0}$ is defined in (3.1), and θ_1 and η are recalled in Appendix B). The $g = e$ case of (1.1) expresses the K3 elliptic genus in terms of $H^{(2)}$, and is the starting point of [18].

This suggests that the Mathieu moonshine module might be realized in terms of a suitably chosen K3 sigma model, but it was found in [26] that the symmetries of these objects are precisely the subgroups of the automorphism group of the Leech lattice—i.e., the *Conway group*, $Co_0 \simeq 2.Co_1$—that fix a 4-space. In particular, M_{24} does not appear. Interestingly, it has been found [14] that suitable trace functions attached to the moonshine module for Conway's group (see [16,13]) attach weak Jacobi forms of weight 0 and index 1 (with level) to 4-space-fixing automorphisms of the Leech lattice, and this construction recovers the K3 elliptic genus when applied to the trivial symmetry. More generally, many—but not all—of the $Z_g^{(2)}$ appear in this way. So the Conway moonshine