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On the spectral radius and energy of the weighted adjacency matrix of a graph[☆]

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ABSTRACT

Let G be a graph of order n and let d_i be the degree of the vertex v_i in G for $i = 1, 2, \ldots, n$. The weighted adjacency matrix A_{db} of G is defined so that its (i, j)-entry is equal to $\frac{d_i+d_j}{d_i d_j}$ if the vertices v_i and v_j are adjacent, and 0 otherwise. The spectral radius ϱ_1 and the energy $\mathcal{E}_{db}$ of the A_{db}-matrix are examined. Lower and upper bounds on ϱ_1 and $\mathcal{E}_{db}$ are obtained, and the respective extremal graphs are characterized.

1. Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$, where n is the *order* and m is the *size* of G. If the vertices v_i and v_j are adjacent, we write $v_i \sim v_j$ or $v_i v_j \in E$. For $i = 1, 2, \ldots, n$, let d_i be the degree of the vertex v_i of G. The maximum and minimum degrees of the graph G are denoted by Δ and δ, respectively.

Given a graph G, the *adjacency matrix* $A = A(G)$ is defined so that its (i, j)-entry is equal to 1 if $v_i v_j \in E$ and 0 otherwise. Note that A is real symmetric. Hence, its eigenvalues are real and can be arranged in non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \lambda_n$ for a connected graph G, where λ_1 is usually referred to as the *spectral radius* of G. For the adjacency spectra, one may be referred to [9,24,26] and the reference therein.

The *energy* of the graph G is defined as

$$\mathcal{E} = \mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1.1}$$

For its basic properties and applications, including lower and upper bounds, see [16–18,22] and the nice monograph [21] on energy of graphs.

In 1994, Yang et al. [29] proposed the *extended adjacency matrix* of graph G, written by $A_{ex}(G)$, which was defined so that its (i, j)-entry is equal to $\frac{1}{2}\left(\frac{d_j}{d_i} + \frac{d_i}{d_j}\right)$ if $v_i \sim v_j$ and 0 otherwise. Note that A_{ex} is a real symmetric matrix of order n, all its eigenvalues are real, which can be denoted by $\eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_n$. Yang et al. [29] also investigated the sum of the absolute

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values of the eigenvalues of the A_{ex}-matrix, which was just the *extended graph energy*, defined as

$$\mathcal{E}_{ex} = \mathcal{E}_{ex}(G) = \sum_{i=1}^{n} |\eta_i|. \tag{1.2}$$

We know from [5] that the extended graph energy $\mathcal{E}_{ex}$ probably was the first and earliest modification of the ordinary (on the adjacency matrix based) graph energy $\mathcal{E}$. It was conceived more than ten years before the Laplacian [11], distance [15], matching [3,28] and Randić [6,7] energies were put forward.

Motivated by [5,29], in this paper we introduce a new degree-based adjacency matrix of the graph G, written by $A_{db}(G)$. It is defined so that its (i, j)-entry is equal to $\frac{d_i+d_j}{d_i d_j}$ if $v_i \sim v_j$ and 0 otherwise. In fact, on the one hand, $A_{db}(G)$ may be viewed as a new type of weighted adjacency matrix. On the other hand, the sum of the inverse for each non-zero entry of A_{db} is just the $2 \cdot ISI(G)$, where

$$ISI(G) = \sum_{ij \in E_G} \frac{1}{\frac{1}{d_i} + \frac{1}{d_j}}.$$

This invariant is called the *inverse sum indeg index*. It was selected in Vukičvić and Gašerov [27] as a significant predictor of total surface area of octane isomers and for which the extremal graphs obtained with the help of MathChem have a particularly simple and elegant structure. The mathematical properties of $ISI(G)$ were extensively studied in [1,8,25,27]. Along this line, it is natural and interesting for us to study the spectral properties of $A_{db}(G)$.

Note that A_{db} is a real symmetric matrix of order n. Hence, all its eigenvalues are real and can be arranged as $\varrho_1 \geqslant \varrho_2 \geqslant \cdots \geqslant \varrho_n$, where the largest eigenvalue ϱ_1 is called the *weighted spectral radius* of the graph G.

The modification of the adjacency graph energy $\mathcal{E}$ [10] motivates our approach to introduce another type of energy, namely the *weighted energy*, for the graph G, which is defined as

$$\mathcal{E}_{db} = \mathcal{E}_{db}(G) = \sum_{i=1}^{n} |\varrho_i|. \tag{1.3}$$

In the later part of this paper we shall need two graph invariants. One is the first Zagreb index M_1 [14,19,20,31] of G, which is defined as

$$M_1 = M_1(G) = \sum_{i=1}^{n} {d_i}^2.$$

The other one is the index M^* of G which is defined as

$$M^* = M^*_G = \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j}.$$

As usual, by $K_{p,q}$ $(p+q=n)$, K_n and $K_{1,n-1}$ we denote, respectively, the complete bipartite graph, the complete graph and the star on n vertices. For other undefined notation and terminology from graph theory and matrix theory, the readers are referred to [2,30].

The rest of the paper is organized as follows. In Section 2, we state some preliminary results, needed for the subsequent sections. In Section 3, we give some lower and upper bounds on the weighted spectral radius and characterize the extremal graphs. In the last section, we obtain some lower and upper bounds on the weighted graph energy and characterize the extremal graphs.

2. Lemmas

In this section, we state some previously known results that are needed in the next two sections.

Lemma 2.1 [29]**.** *If C is a real symmetric $n \times n$ matrix with eigenvalues $\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_n$, then for any $\mathbf{x} \in \mathbf{R}^n$, such that $x \neq 0$,*

$$\mathbf{x}^T C \mathbf{x} \leqslant \xi_1 \mathbf{x}^T \mathbf{x}.$$

Equality holds if and only if $\mathbf{x}$ is an eigenvector of C corresponding to the largest eigenvalue ξ_1.

Lemma 2.2 [13]**.** *Let $C = (c_{ij})$ and $D = (d_{ij})$ be real symmetric, non-negative matrices of order n. If $C \geqslant D$, i.e., $c_{ij} \geqslant d_{ij}$ for all i, j, then $\xi_1(C) \geqslant \xi_1(D)$, where ξ_1 is the largest eigenvalue.*

Lemma 2.3 [23]**.** *Let C be a real symmetric matrix of order n, and let C_k be its leading $k \times k$ submatrix. Then, for $i = 1, 2, \ldots, k$,*

$$\xi_{n-i+1}(C) \leqslant \xi_{k-i+1}(C_k) \leqslant \xi_{k-i+1}(C),$$

where $\xi_i(C)$ is the ith largest eigenvalue of C.

Lemma 2.4 [12]**.** *Let G be a connected graph of order n with m edges. Then*

$$\lambda_1(G) \leqslant \sqrt{2m - n + 1}$$