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Spectral analogues of Erdős' theorem on Hamilton-connected graphs

Jia Wei[a], Zhifu You[b], Hong-Jian Lai[c,*]

[a] Department of Mathematics, South China University of Technology, Guangzhou 510640, PR China
[b] School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou 510665, PR China
[c] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

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ABSTRACT

A graph G is Hamilton-connected if for any pair of vertices v and w, G has a spanning (v, w)-path. Extending theorems of Dirac and Ore, Erdős prove a sufficient condition in terms of minimum degree and the size of G to assure G to be Hamiltonian. We investigate the spectral analogous of Erdős' theorem for a Hamilton-connected graph with given minimum degree, and prove that there exist two graphs $\{L_n^k, M_n^k\}$ such that each of the following holds for an integer $k \geq 3$ and a simple graph G on n vertices.

(i) If $n \geq 6k$, $\delta(G) \geq k$, and $|E(G)| > \binom{n-k}{2} + k(k+1)$, then G is Hamilton-connected if and only if $C_{n+1}(G) \notin \{L_n^k, M_n^k\}$.

(ii) If $n \geq \max\{6k, \frac{1}{2}k^3 - \frac{1}{2}k^2 + k + 4\}$, $\delta(G) \geq k$ and spectral radius $\lambda(G) \geq n - k$, then G is Hamilton–connected if and only if $G \notin \{L_n^k, M_n^k\}$.

1. Introduction

We consider finite and simple graphs, with undefined notation and term following [3]. We normally use $e(G)$, n, $\delta(G)$ and $A(G)$ to denote $|E(G)|$, $|V(G)|$, the minimum degree and the adjacency matrix of a graph G, respectively. The largest eigenvalue of $A(G)$, called *the spectral radius* of G, is denoted by $\lambda(G)$. Let H be a subgraph of a graph G, and let $u \in V(G)$. The set of neighbors of a vertex u in H is denoted by $N_H(u)$. Thus

$$N_H(u) = \{v \in V(H) : uv \in E(G)\}.$$

Define $d_H(u) = |N_H(u)|$. A *clique* is a subset of vertices of an undirected graph whose induced subgraph is a complete graph. The maximum size of a clique of a graph is called *clique number*, denoted by $\omega(G)$. For $S \subseteq V(G)$, the *induced subgraph* $G[S]$ is the graph with vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$.

The disjoint union of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The disjoint union of k copies of a graph G is denoted by kG. *The join* of G_1 and G_2, denoted by $G_1 \vee G_2$, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy | x \in V(G_1), y \in V(G_2)\}$.

A path (or a cycle, respectively) of a graph G is called a *Hamilton path (or Hamilton cycle, respectively)* if it passes through all the vertices of G. A graph is *Hamilton-connected* if any two vertices are connected by a Hamilton path. The investigation of hamiltonian graphs has a long history. Dirac and Ore proved the following.

* Corresponding author.
E-mail addresses: 201520122177@mail.scut.edu.cn (J. Wei), youzhf@hotmail.com (Z. You), hjlai@math.wvu.edu (H.-J. Lai).

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Theorem 1.1. *Let G be a graph of order n.*

(i) *(Dirac [6]) If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*
(ii) *(Ore [14]) If $e(G) > \binom{n-1}{2} + 1$, then G is Hamiltonian.*

Motivated by these results, Erdős [7] later extended Theorem 1.1 (ii) by utilizing the minimum degree as a new parameter.

Theorem 1.2. *(Erdős [7]) Let G be a graph of order n and the minimum degree δ and k be an integer with $1 \leq k \leq \delta \leq \frac{n-1}{2}$. If*

$$e(G) > \max\left\{\binom{n-k}{2} + k^2, \binom{\lceil \frac{n+1}{2} \rceil}{2} + \left\lfloor \frac{n+1}{2} \right\rfloor^2\right\},$$

then G is Hamiltonian.

How many edges can ensure a graph to be Hamilton–connected with a given number of vertices? In 1963, Ore [15] answered the question.

Theorem 1.3. *[15] Let G be a graph of order n, if*

$$e(G) \geq \binom{n-1}{2} + 3,$$

then G is Hamilton-connected.

Theorem 1.4. *([16], Theorem 1.8) Let G be a graph of order $n \geq 6k^2 - 8k + 5$ with $\delta(G) \geq k \geq 2$. If $e(G) \geq \frac{n^2-(2k-1)n+2k-2}{2}$, then G is Hamilton-connected unless $cl_{n+1}(G) = K_2 \vee (K_{n-k-1} \cup K_{k-1})$ or $cl_{n+1}(G) = K_k \vee (K_{n-2k-1} \cup \overline{K}_{k-1})$.*

Theorem 1.5. *([16], Corollary 1.10) Let G be a graph of order $n \geq \max\{6k^2 - 8k + 5, \frac{k^3-k^2+4k-1}{2}\}$ with $\delta(G) \geq k \geq 2$. If $\rho(G) \geq n-k$, then G is Hamilton-connected unless $G = K_2 \vee (K_{n-k-1} \cup K_{k-1})$ or $G = K_k \vee (K_{n-2k+1} \cup \overline{K}_{k-1})$.*

The results above, as well as the recent advances in [9,13,16], motivate the current research. In this paper, we present a spectral analogous of Erdős theorem for a Hamilton-connected graph with a given minimum degree. For a graph G, notice that $\delta(G) \geq 3$ is a necessary condition for G to be Hamilton-connected. A sufficient condition for a Hamilton-connected graph in terms of spectral radius is also justified. This paper is independently research work with Chen and Zhang's ([16]) results.

Throughout this paper, for $2 \leq k \leq \frac{n}{2}$, let

$$L_n^k = K_2 \vee (K_{n-k-1} + K_{k-1}) \text{ and } M_n^k = K_k \vee (K_{n-2k+1} + (k-1)K_1).$$

In Section 2, extremal sizes of graphs to ensure Hamilton-connectedness are investigated. These will be applied in Section 3 to find an optimal spectral sufficient condition for a graph G to be Hamilton-connected.

2. Extremal sizes of Hamilton-connected graphs

Let X, Y be vertex subsets of a graph G. Following [3], we adopt these notation: $e(X) = |E(G[X])|$,

$$E_G[X, Y] = \{xy \in E(G) : x \in X \text{ and } y \in Y\}, \text{ and } e(X, Y) = |E_G[X, Y]|.$$

Throughout this section, if J is a subgraph of G and $v \in V(G) - V(J)$, define $d_J(v) = |E_G[\{v\}, V(J)]|$.

The purpose of this section is to prove two extremal results, namely, Theorems 2.2 and 2.5 in this section, on the optimal sizes to assure a graph to be Hamilton-connected . We state some known results as our tools.

Theorem 2.1. *(Erdős, Gallai, [8]) Let G be a graph of order $n \geq 3$, and u, v are any pair distinct and nonadjacent vertices. If*

$$d_G(u) + d_G(v) \geq n + 1,$$

then G is Hamilton-connected.

Lemma 2.1. *[1] Let G be a graph of order $n \geq 3$ with the degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $2 \leq t \leq \frac{n}{2}$ such that $d_{t-1} \leq t$ and $d_{n-t} \leq n-t$, then G is Hamilton-connected.*

Theorem 2.2. *Let G be a graph with order n and the minimum degree δ, and let k be an integer with $2 \leq k \leq \delta$. If*

$$e(G) > \max\left\{\binom{n-k+1}{2} + k(k-1), \binom{\lceil \frac{n}{2} \rceil + 1}{2} + \left\lfloor \frac{n}{2} \right\rfloor\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\right\}, \tag{2.1}$$

then G is Hamilton-connected.

Proof. Suppose that G is not Hamilton-connected. By Lemma 2.1, there exists an integer t such that $d_{t-1} \leq t$, where $k \leq t \leq \frac{n}{2}$. Without loss of generality, let $d(v_i) = d_i$ for $1 \leq i \leq t-1$. The number of edges which are not incident to any vertex in