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Clenshaw–Curtis-type quadrature rule for hypersingular integrals with highly oscillatory kernels☆

Guidong Liu, Shuhuang Xiang*

School of Mathematics and Statistics, Central South University, Yue Lu Qu, Changsha Hunan 410083 , China

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ABSTRACT

The Clenshaw–Curtis-type quadrature rule is proposed for the numerical evaluation of the hypersingular integrals with highly oscillatory kernels and weak singularities at the end points $⨍_{-1}^{1}\frac{(x+1)^{\alpha}(1-x)^{\beta}g(x)}{(x-s)^{m}}e^{ikx}\,\mathrm{d}x,\ \ s\in(-1,1)$ for any smooth functions $g(x)$. Based on the fast Hermite interpolation, this paper provides a stable recurrence relation for these modified moments. Convergence rates with respect to the frequency k and the number of interpolation points N are considered. These theoretical results and high accuracy of the presented algorithm are illustrated by some numerical examples.

1. Introduction

The boundary element method is one of the most frequently used numerical approaches for solving partial differential equations (PDEs) arose in many mathematical, physical and engineering problems [7,24,29,32]. It leads the two-dimensional PDEs to the one-dimensional Fredholm integral equations of the form

$$\lambda u(s)+⨍_{a}^{b}\frac{K(s,x)}{(x-s)^{m}}u(x)\,\mathrm{d}x=f(s),\quad s\in(a,b),\quad m=1,2,\ldots,\tag{1.1}$$

where λ is a scalar, $u(x)$ is the unknown function and $f(s)$ is a given function. The integral in (1.1) is understood as the Cauchy principal value for $m=1$ and Hadamard finite part for $m\geq 2$ [39,54].

Much work has focused on the numerical solution for (1.1) with constant kernel $K(s,x)=1$. Due to the strong singularity, the solution can be represented as $u(x)=(1+x)^{\alpha}(1-x)^{\beta}g(x):=\omega(x)g(x)$ with $\alpha,\beta\in(-1,1)$ and $g(x)$ being a smooth function on $[a, b]$ [3,6,11,13,20]. However, in many fields, such as the electromagnetic scattering and quantum mechanism, the kernel functions are usually highly oscillatory with $K(s,x)=e^{ik(x-s)}$, $k\gg 1$ [2,5,26,36], which leads the integral involved to

$$\mathcal{I}(g,s,\alpha,\beta,k)=⨍_{-1}^{1}\frac{\omega(x)e^{ikx}}{(x-s)^{m}}g(x)\,\mathrm{d}x,\quad s\in(-1,1),\quad -1<\alpha,\beta<1,\tag{1.2}$$

where without loss of generality, the interval $[a, b]$ has been transformed to $[-1, 1]$.

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* Corresponding author.
E-mail addresses: csu_guidongliu@csu.edu.cn (G. Liu), xiangsh@csu.edu.cn (S. Xiang).

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Numerical computation of (1.2) has been well studied in the case of $k = 0$ and $\omega(x) \equiv 1$, for example, the (composite) Newton-Cotes method [33,35,51] , the Gauss-type method [10,14,30,31,38,47,48] and other approaches [17]. In fact, these methods can be extended to the general cases $\omega(x) \not\equiv 1$. A typical approach is splitting the integrand into a singular part and a regular part as follows

$$\mathcal{I}(g, s, \alpha, \beta, 0) = \int_{-1}^{1} \omega(x) \frac{g(x) - \sum_{j=0}^{m-1} \frac{g^{(j)}(s)}{j!}(x-s)^j}{(x-s)^m} \, \mathrm{d}x + \sum_{j=0}^{m-1} \frac{g^{(j)}(s)}{j!} \fint_{-1}^{1} \frac{\omega(x)}{(x-s)^{m-j}} \, \mathrm{d}x, \tag{1.3}$$

where the first integral can be approximated by some ordinary quadrature rules, such as Gaussian, Fejér or Clenshaw–Curtis quadrature rule [31]. While the second part can be evaluated exactly by Erdogan et al. [19]

$$\fint_{-1}^{1} \frac{\omega(x)}{x-s} \, \mathrm{d}x = \pi \omega(s) \cot \pi \beta - 2^{\alpha+\beta} \frac{\Gamma(\beta)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} {}_2F_1\left(1, -\alpha-\beta; 1-\beta; \frac{1-s}{2}\right), \tag{1.4}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function and its derivatives are [1]

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} {}_2F_1(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} {}_2F_1(a+n, b+n; c+n; z), \quad (a)_n = a(a+1)\cdots(a+n-1). \tag{1.5}$$

Nevertheless, this technique can not be applied to the case $k \gg 1$ since the quadrature rules will suffer from large number of k when approximating the regular part in the right hand side of (1.3).

In the case $k \gg 1$ and $\omega(x) \equiv 1$ in (1.2), Xiang et al. [54] studied the uniform approximation scheme for (1.2). The principle is separating the integrand into oscillating and regular parts, and approximating the regular part by the Chebyshev interpolation. In [21], Fang proposed a steepest descent method for the Cauchy principal value of (1.2) ($m = 1$), which requires the analyticity of $g(x)$ in a large complex region. Other works on the computation of highly oscillatory integrals with algebraic and Cauchy-type singularities are well studied, we refer the readers to [4,16,25,27,40,49,50,55].

However, all these numerical methods can not be applied to (1.2) directly due to the existence of oscillation and weak singularities at the end points. In this paper, we present a Clenshaw–Curtis-type quadrature rule for the computation of these hypersingular integrals (1.2), particularly for $k \gg 1$, which is based on the fast Hermite interpolation schemes and the stable recurrence relation for the modified moments defined in (2.6). Theoretical analysis and numerical experiments show the efficiency and accuracy.

The rest of this paper is organized as follows. In Section 2, we describe the Clenshaw–Curtis-type quadrature algorithm for the integral (1.2). The fast implementation of the Hermite interpolation and the recurrence relation for the modified moments are presented. In Section 3, the error estimate of the proposed algorithm is given and shows explicitly how it depends on the parameters k and N. In Section 4, the stability of the recursion (2.15) is proved. Finally, these theoretical results are illustrated by some numerical examples in Section 5.

2. Clenshaw–Curtis-type quadrature rule

2.1. Description of the algorithm

The Clenshaw–Curtis quadrature rule has been extensively studied in [8,22,44,53,54], which interpolates $g(x)$ at the Clenshaw–Curtis point set $\boldsymbol{X}_{N+1} = \left\{ x_j = \cos \frac{j\pi}{N} \right\}_{j=0}^{N}$ in terms of

$$g(x) \approx P_N(x) := \sum_{n=0}^{N}{}'' c_n T_n(x), \tag{2.1}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind, the double prime denotes a summation whose first and last terms are halved, and the coefficients

$$c_n = \frac{2}{N} \sum_{j=0}^{N}{}'' g(x_j) T_n(x_j) \tag{2.2}$$

can be implemented by FFT in $\mathcal{O}(N \log N)$ operations [9,23,44]. Many numerical results can be found in [45,54].

In this paper, we consider a new quadrature rule, Clenshaw–Curtis-type quadrature rule, for the integral (1.2), which approximates the integrand by a Hermite interpolation of the form

$$\widehat{P}(x_j) = g(x_j), \;\; j = 0, \ldots, N; \quad \widehat{P}^{(j)}(s) = g^{(j)}(s), \;\; j = 0, \ldots, m-1. \tag{2.3}$$

For any fixed s, we choose N such that $s \notin \boldsymbol{X}_{N+1}$ and rewrite the Hermite interpolant as a Chebyshev series

$$\widehat{P}_{N+m}(x) = \sum_{n=0}^{N+m} b_n T_n(x). \tag{2.4}$$