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Star edge-coloring of graphs with maximum degree four

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ABSTRACT

The star chromatic index $\chi'_{st}(G)$ of a graph G is the smallest integer k for which G has a proper k-edge-coloring without bichromatic paths or cycles of length four. In this paper, we prove that (1) if G is a graph with $\Delta = 4$, then $\chi'_{st}(G) \le 14$; and (2) if G is a bipartite graph with $\Delta = 4$, then $\chi'_{st}(G) \le 13$.

1. Introduction

Only simple graphs are considered in this paper. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and minimum degree of G, respectively. If no ambiguity arises in the context, $\Delta(G)$ is simply written as Δ. Two edges $e_1, e_2 \in E(G)$ are said to be *distance one* if they are adjacent in G, and *distance two* if they are not adjacent, but there exists an edge $e' \in E(G)\backslash\{e_1, e_2\}$ adjacent to both of them.

A *proper k-edge-coloring* of a graph G is a mapping $\phi : E(G) \to \{1, 2, \dots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. The *chromatic index* $\chi'(G)$ of G is the smallest integer k such that G has a proper k-edge-coloring. Similarly, we may define the (vertex) chromatic number $\chi(G)$ of a graph G.

A proper k-edge-coloring ϕ of a graph G is called a *strong k-edge-coloring* if any two edges of distance at most two get distinct colors. That is, every color class is an induced matching in G. The *strong chromatic index* of G, denoted $\chi'_s(G)$, is the smallest integer k such that G has a strong k-edge-coloring.

A proper k-edge-coloring ϕ of a graph G is called a *star k-edge-coloring* if there do not exist bichromatic paths or cycles of length four. Namely, at least three colors are required to color every path and cycle of length four. The *star chromatic index* of G, denoted $\chi'_{st}(G)$, is the smallest integer k such that G has a star k-edge-coloring.

As an easy observation, it holds trivially that $\chi'_s(G) \ge \chi'_{st}(G) \ge \chi'(G) \ge \Delta$.

The strong edge-coloring of graphs was introduced by Fouquet and Jolivet [5]. Erdős and Nešetřil raised the following conjecture and showed that the upper bounds are tight:

Conjecture 1. *For a graph G,*

$$\chi'_s(G) \le \begin{cases} 1.25\Delta^2, & \text{if } \Delta \text{ is even;} \\ 1.25\Delta^2 - 0.5\Delta + 0.25, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Using probabilistic method, Molloy and Reed [14] showed that $\chi'_s(G) \le 1.998\Delta^2$ when G has sufficiently large Δ. This result was further improved in [3] to that $\chi'_s(G) \le 1.93\Delta^2$.

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In 2008, Liu and Deng [13] first investigated the star edge-coloring of graphs by showing that $\chi'_{st}(G) \le \lceil 16(\Delta-1)^{\frac{3}{2}} \rceil$ for any graph G with $\Delta \ge 7$. In 2013, Dvořák, et al. [4] discussed the star chromatic indices of complete graphs and subcubic graphs. The partial results obtained and an open problem are stated as follows:

Theorem 1 [4]**.** *Let G be a subcubic graph. Then*

(1) $\chi'_{st}(G) \le 7$.
(2) *If G is cubic, then* $\chi'_{st}(G) \ge 4$*; and the equality holds if and only if G covers the graph of 3-cube.*

Conjecture 2 [4]**.** *Every subcubic graph G has* $\chi'_{st}(G) \le 6$.

As observed in [4], the star chromatic index of $K_{3,3}$ is 6. Hence the upper bound 6 in Conjecture 2 is the best possible.

Bezegová et al. [1] showed that every tree T has $\chi'_{st}(T) \le \lfloor 1.5\Delta \rfloor$ and demonstrated that the upper bound $\lfloor 1.5\Delta \rfloor$ is tight. In the same paper [1], the authors also verified that every outerplanar graph G satisfies $\chi'_{st}(T) \le \lfloor 1.5\Delta \rfloor + 12$ and conjectured that the constant 12 can be replaced by 1. Very recently, the authors of this paper [17] made use of the edge-partition technique to improve this result and to obtain some better bounds on the star chromatic index of planar graphs.

Theorem 2 [17]**.** *Let G be a planar graph. Then*

(1) $\chi'_{st}(G) \le 2.75\Delta + 18$.
(2) $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 18$ *if G does not contain 4-cycles.*
(3) $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 13$ *if the girth of G is at least 5.*
(4) $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 3$ *if the girth of G is at least 8.*
(5) $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 5$ *if G is outerplanar.*
(6) $\chi'_{st}(G) \le 2.25\Delta + 6$ *if G is* K_4*-minor free.*

For other results on the star edge-coloring of graphs can be seen in [7–9,11,12].

Conjecture 1 asserts that every graph G with $\Delta = 4$ has $\chi'_s(G) \le 20$. The currently best known upper bound is that $\chi'_s(G) \le 21$, due to Huang, Santana and Yu [6]. In this paper, we focus on the star edge-coloring of graphs with maximum degree four. More precisely, we will show that every graph G with $\Delta = 4$ has $\chi'_{st}(G) \le 14$; and moreover $\chi'_{st}(G) \le 13$ if such graph G is bipartite.

2. Preliminary results

Before proving our main results, we need to introduce a few concepts and preliminary results. An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for all $i \ne j$.

Lemma 1 [16]**.** *If G is a* $2k$*-regular graph with* $k \ge 1$*, then G is 2-factorizable.*

Suppose that H is a subgraph of a graph G. A *restricted-strong k-edge-coloring* of H on G is a mapping $\phi : E(H) \to \{1, 2, \ldots, k\}$ such that any two edges $e_1, e_2 \in E(H)$ of distance at most two in G get distinct colors. The *restricted-strong chromatic index* of H on G, denoted $\chi'_s(H|_G)$, is the smallest integer k such that H has a restricted-strong k-edge-coloring on G.

The following useful inequality was established in [17]:

Lemma 2. *If a graph G can be edge-partitioned into two graphs F and H, then*

$$\chi'_{st}(G) \le \chi'_{st}(F) + \chi'_s(H|_G).$$

By an easy observation, we conclude that the star chromatic index of graphs possesses the following hereditary:

Lemma 3. *If H is a subgraph of a graph G, then* $\chi'_{st}(H) \le \chi'_{st}(G)$.

An *edge list L* for a graph G is a mapping that assigns a set of colors to each edge of G. Given an edge list L for a graph G, we say that G is star L-edge-colorable if it has a star edge-coloring ϕ such that $\phi(e) \in L(e)$ for every edge e of G. The *list star chromatic index*, denoted $\mathrm{ch}'_{st}(G)$, of a graph G is the least integer k such that for every edge list L of G with $|L(e)| = k$ for every $e \in E(G)$, G is star L-edge-colorable.

The following Lemma 4 is easily confirmed by a simple discussion.

Lemma 4. *Let* P_n *be a path on order* $n \ge 2$*. Then*

$$\mathrm{ch}'_{st}(P_n) = \chi'_{st}(P_n) = \begin{cases} 1, & \text{if } n = 2; \\ 2, & \text{if } 3 \le n \le 4; \\ 3, & \text{if } n \ge 5. \end{cases}$$

Lemma 5. *Let* C_n *be a cycle on order* $n \ge 3$*. Then*

$$\mathrm{ch}'_{st}(C_n) = \chi'_{st}(C_n) = \begin{cases} 3, & \text{if } n \ne 5; \\ 4, & \text{if } n = 5; \end{cases}$$