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Stokes phenomenon, Gelfand–Zeitlin systems and relative Ginzburg–Weinstein linearization

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ABSTRACT

In 2007, Alekseev–Meinrenken proved that there exists a Ginzburg–Weinstein diffeomorphism from the dual Lie algebra $\mathfrak{u}(n)^*$ to the dual Poisson Lie group $U(n)^*$ compatible with the Gelfand–Zeitlin integrable systems. In this paper, we explicitly construct such diffeomorphisms via Stokes phenomenon and Boalch's dual exponential maps. Then we introduce a relative version of the Ginzburg–Weinstein linearization motivated by irregular Riemann–Hilbert correspondence, and generalize the results of Enriquez–Etingof–Marshall to this relative setting. In particular, we prove the connection matrix for a certain irregular Riemann–Hilbert problem satisfies a relative gauge transformation equation of the Alekseev–Meinrenken dynamical r-matrices. This gauge equation is then derived as the semiclassical limit of the relative Drinfeld twist equation.

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1. Introduction and main results

The Ginzburg–Weinstein linearization theorem [24] states that for any compact Lie group K with its standard Poisson structure, the dual Poisson Lie group K^* is Poisson isomorphic to the dual of the Lie algebra $\mathfrak{k}^*$, with its canonical linear (Kostant–Kirillov–Souriau) Poisson structure. When $K = U(n)$, the Poisson manifolds $\mathrm{u}(n)^*$ and $U(n)^*$ carry more structures: Guillemin–Sternberg [25] introduced the Gelfand–Zeitlin integrable system on $\mathrm{u}(n)^*$; later on, Flaschka–Ratiu [22] described a multiplicative Gelfand–Zeitlin system for the dual Poisson Lie group $U(n)^*$. Then in [4], Alekseev and Meinrenken constructed a Ginzburg–Weinstein linearization which intertwines the Gelfand–Zeitlin systems on $\mathrm{u}(n)^*$ and $U(n)^*$. Their diffeomorphism is unique with the extra conditions that the derivative at 0 is identity and that it sends the real part (real symmetric matrices) of $\mathrm{u}(n)^*$ to the real part of $U(n)^*$.

There are various generalizations of Ginzburg–Weinstein linearization to the complex and formal setting. In [8], Boalch pointed out that, for $G = \mathrm{GL}_n(\mathbb{C})$ equipped with the standard Poisson Lie group structure, the dual Poisson Lie group G^* is identified with a moduli space of meromorphic connections with certain irregular singularity. This viewpoint enabled him to define a class of "dual exponential maps" $\nu : \mathfrak{g}^* \to G^*$ by taking the Stokes data of the meromorphic connections. Then his remarkable result shows that these (irregular Riemann–Hilbert) maps $\nu : \mathfrak{g}^* \to G^*$ are local Poisson isomorphisms. Later on in [10], this result, along with the definition of such Stokes data, was extended beyond $\mathrm{GL}_n(\mathbb{C})$ to any complex reductive Lie groups. As a consequence (see [8] Section 5 and [10] Theorem 8), the dual exponential maps of Boalch give a natural family of Ginzburg–Weinstein isomorphisms. This was before the work of Alekseev–Meinrenken [4] and works for any compact group. On the other hand, Enriquez–Etingof–Marshall [17] constructed formal Poisson isomorphisms between the formal Poisson manifolds $\mathfrak{g}^*$ and G^*. Their result relies on constructing a formal map $\rho : \mathfrak{g}^* \to G$ satisfying a vertex-IRF gauge transformation equation [19] of the Alekseev–Meinrenken r-matrix [3]. Furthermore, one solution of this gauge equation is derived as the semiclassical limit of a Drinfeld twist. Ginzburg–Weinstein linearization in more general setting can be found in [5].

The comparison of the above two approaches of Boalch and Enriquez–Etingof–Marshall enables us to unveil some unexpected relations between Stokes phenomenon, dynamical r-matrices and Drinfeld twists [36]. These mysterious relations are later on understood in our joint work with Toledano Laredo [34] by studying the Stokes phenomenon of dynamical Knizhnik–Zamolodchikov equations [21]. However, the possible