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Differential algebra of cubic planar graphs

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ABSTRACT

In this article we associate a combinatorial differential graded algebra to a cubic planar graph G. This algebra is defined combinatorially by counting binary sequences, which we introduce, and several explicit computations are provided. In addition, in the appendix by K. Sackel the $\mathbb{F}_q$-rational points of its graded augmentation variety are shown to coincide with $(q+1)$-colorings of the dual graph.

1. Introduction

This article defines new algebraic structures associated to cubic graphs. The inspiration for our construction comes from symplectic field theory [9,11] and the theory of constructible sheaves [20,23]. To every planar cubic graph, we assign to it a differen-

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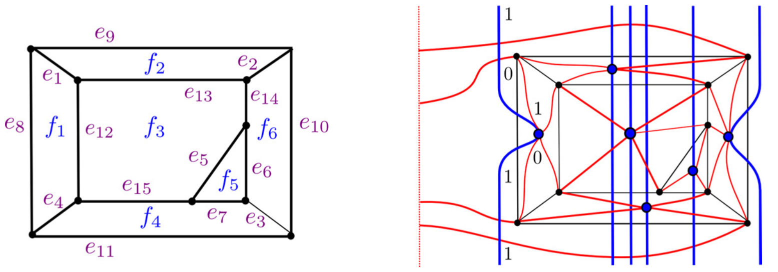

Fig. 1. Planar cubic graph G (left) and its decoration (right). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

tial graded algebra, which describes a number of combinatorial features of a graph. In particular, the augmentation variety of this algebra recovers the chromatic data of the dual graph, established in Appendix A, written by K. Sackel. In terms of graph Legendrians, this can be interpreted as the algebraic part of the conjectural correspondence between the augmentation variety of the Legendrian contact homology algebra and the moduli space of rank-1 constructibles sheaves [22,25], which also constitute part of the developing program for mirror symmetry [5,6].

The combinatorics presented in this work lie at the intersection of three fields: the contact topology of Legendrian surfaces [3,7], the combinatorics of cubic graphs [8,24] and the enumerative geometry of open Gromov–Witten invariants [4,12]. In addition, the structures we introduce have some interesting connections to spectral networks [14,15]: the binary sequences arising in our construction distinguish a special type of framed 2d-4d BPS states of the supersymmetric $N = 2$ 4d-theories of class S associated to the Lie algebra $\mathfrak{g} = su(2)$. These connections will be explored in more depth in a later work.

For now, this article defines and explores this algebraic structure from a purely combinatorial perspective. Given a planar cubic graph G with $2g + 2$ vertices and a ground field $\mathbb{F}$, we consider the base ring $\Lambda_G = \mathbb{F}[e_1^{\pm 1}, \ldots, e_{3g+3}^{\pm 1}]$ of Laurent polynomials in the set of edges, and the unital Λ_G-algebra generated by the faces of G and three additional generators:

$$\widetilde{\mathcal{A}}_G = \langle x, y, z, f_j : 1 \leq j \leq g + 2\rangle.$$

The goal is to endow $\widetilde{\mathcal{A}}_G$ with the structure of a dg- algebra which contains interesting information about the graph, and in particular show that it contains the chromatic polynomial. Here is an example: consider the modified 4-prism graph in the left of Fig. 1. We equip the graph G with blue and red paths as in the right of Fig. 1, together we denote the choice of these decorations by Γ.

Then the count of binary sequences along the blue paths interacting with the red paths allows us to define a differential operator $\widetilde{\partial}_\Gamma$ on $\widetilde{\mathcal{A}}_G$, and in particular the red paths shall dictate the differential on the faces $F = \{f_1, \ldots, f_{g+2}\}$. For instance, in the case of Fig. 1 above the differential of x contains the information of the binary sequence