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Vortex sheets and diffeomorphism groupoids

Anton Izosimov [a], Boris Khesin [b,*]

[a] *Department of Mathematics, University of Arizona, United States of America*
[b] *Department of Mathematics, University of Toronto, Canada*

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A B S T R A C T

In 1966 V. Arnold suggested a group-theoretic approach to ideal hydrodynamics in which the motion of an inviscid incompressible fluid is described as the geodesic flow of the right-invariant L^2-metric on the group of volume-preserving diffeomorphisms of the flow domain. Here we propose geodesic, group-theoretic, and Hamiltonian frameworks to include fluid flows with vortex sheets. It turns out that the corresponding dynamics is related to a certain groupoid of pairs of volume-preserving diffeomorphisms with common interface. We also develop a general framework for Euler–Arnold equations for geodesics on groupoids equipped with one-sided invariant metrics.

Contents

* Corresponding author.
E-mail addresses: izosimov@math.arizona.edu (A. Izosimov), khesin@math.toronto.edu (B. Khesin).

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1. Introduction

Vortex sheets are hypersurfaces of discontinuity in fluid velocity with different speed of fluid layers on different sides of the hypersurface. They naturally appear, e.g., in the flow past an airplane wing [6]. In this paper we develop geodesic, group-theoretic, and Hamiltonian frameworks for their description.

In 1966 V. Arnold proved that the Euler equation for an ideal fluid describes the geodesic flow of a right-invariant metric on the group of volume-preserving diffeomorphisms of the flow domain [1]. This insight turned out to be indispensable for the study of Hamiltonian properties and conservation laws in hydrodynamics, fluid instabilities, topological properties of flows, as well as a powerful tool for obtaining sharper existence and uniqueness results for Euler-type equations [2]. However, the scope of applicability of Arnold's approach is limited to systems whose symmetries form a Lie group. At the same time, there are many problems in fluid dynamics, such as free boundary problems or (discontinuous) fluid flows with vortex sheets, whose symmetries should instead be regarded as a groupoid: e.g., only those of the maps corresponding to fluid configurations with moving boundary admit composition, for which the image of one map coincides with the source of the other.

In this paper we propose a strategy to extend Arnold's framework to Lie groupoids and develop a groupoid-theoretic description for incompressible fluid flows with vortex sheets, i.e., flows whose velocity field has a jump discontinuity along a hypersurface. It turns out