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Material tailoring for reducing stress concentration factor at a circular hole in a functionally graded material (FGM) panel

G.J. Nie[a,b,*], Z. Zhong[a], R.C. Batra[b]

[a] School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, China
[b] Department of Biomedical Engineering and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

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ABSTRACT

By assuming that Young's modulus and Poisson's ratio of a linearly elastic and isotropic material vary along the radial direction in a panel with a circular hole and deformed by a far field uniaxial tensile traction, we first *analytically* find the stress concentration factor, K, at the hole. The problem is solved by superposing solutions of two problems – one of uniform biaxial tension and the other of pure shear. The solutions of the first and the second problem are, respectively, in terms of hypergeometric functions and Frobenius series. Subsequently, we *analytically* study the material tailoring problem for uniform biaxial tension, and give explicit variation of Young's modulus to achieve a prespecified K. For the panel loaded by a far field uniaxial tensile traction, we show that the K can be reduced by a factor of about 8 by appropriately grading Young's modulus and Poisson's ratio in the radial direction. By plotting K versus the two inhomogeneity parameters, we solve the material tailoring problem for a panel loaded with a far field uniaxial traction. The analytical results should serve as benchmarks for verifying the accuracy of approximate/numerical solutions for an inhomogeneous panel.

1. Introduction

Even though the mechanical behavior of an inhomogeneous material has been studied since 1950's, there has been tremendous activity in this field during the last three decades [1–6]. A heterogeneous material with continuous spatial variation of material parameters is often called a functionally graded material (FGM). With the availability of 3-D printing for manufacturing materials with complex microstructures, it is now feasible to fabricate structures to have the optimum stress and strain distributions for enhancing their mechanical properties under prescribed loads [7,8]. One such problem is controlling the stress concentration factor, K, around a circular hole in a panel.

It is well known that K at a circular hole in an infinite panel composed of a homogeneous, isotropic and linearly elastic material deformed in uniaxial tension equals 3 [9]. For orthotropic materials Lekhnitskii et al. [10] have deduced K for an infinite plate containing a circular hole and deformed by remote uniaxial tensile tractions. Based on Lekhnitskii's solution of the plane elastostatics problem using complex variables, Britt [11], Tenchev et al. [12] and Xu et al. [13,14], respectively, found K for anisotropic rectangular panels with centrally located circular and elliptical cutouts, laminated composites with circular holes, and composite laminates with either an elliptical hole or multiple holes.

Authors of Refs. [15–18] employed the finite element method (FEM) to evaluate stresses in composite laminates with circular holes. Kubair and Bhanu-Chandar [19] and Enab [20] using the FEM found that K is significantly influenced by the spatial variation of the material inhomogeneity. One needs a very fine mesh near the hole and conduct convergence studies to deduce reasonably accurate values of K that can be an arduous task.

Huang and Haftka [21], and Cho and Rowlands [22,23] optimized fiber orientations near a hole to minimize K and increase the load-carrying capacity of composite laminates. Lopes et al. [24] and Gomes et al. [25] found fiber orientation angles and their volume fractions either to minimize the peak stress around cutouts or to maximize the buckling and the first-ply failure load of composite panels. They pointed out that the optimum variable-stiffness designs with a central hole can have nearly the same initial buckling loads as panels with the same volume fractions of fibers but no hole.

By dividing an inhomogeneous material panel into a series of piece-wise homogeneous layers and using the method of complex variables, Yang et al. [26,27], Yang and Gao [28] and Kushwaha and Saini [29] have shown that when Young's modulus decreases with the distance from the hole boundary, $K > 3$. Mohammadi et al. [30] analytically found K around a circular hole in an infinite FGM plate subjected to uniform biaxial tension and pure shear by assuming that both Young's

* Corresponding author at: School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, China.
E-mail address: ngj@tongji.edu.cn (G.J. Nie).

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modulus and Poisson's ratio vary exponentially in the radial direction. By assuming that Young's modulus has a power law variation in the radial direction and Poisson's ratio is a constant, Sburlati [31] studied the elastic response of an FGM annular ring inserted in a hole of a homogeneous plate. Kubair [32] used the method of separation of variables to find closed-form expressions for stresses and displacements in FGM plates with and without holes under anti-plane shear loading and used a non-traditional definition of K.

We note that there are a limited number of analytical studies on the stress concentration around a circular hole in isotropic FGM panels. Furthermore, there are no results on material tailoring for reducing K. We analytical (i) find K at a circular hole in an isotropic FGM panel under a far field uniaxial tensile traction, (ii) analyze the material tailoring problem for uniform biaxial tension loading, and (iii) investigate the effect of material inhomogeneity parameters on K. For far field uniform tensile loading, we provide a response function to estimate inhomogeneity parameters for a desired value of K.

The rest of the paper is organized as follows. Sections 2 and 3, respectively, give the formulation and the solution of the direct problem in which we analyze deformations of the panel under prescribed far field surface tractions. Section 3 is divided into three subsections that provide details of deformations under uniform biaxial tension, pure shear and uniaxial tension, respectively. In Section 4 we analytically solve the material tailoring problem for uniform biaxial tension loading. Section 5 provides numerical results that establish the accuracy and the convergence of the series solution for the pure shear loading, and delineate effects of the variation of the material properties on K and stress distributions. Conclusions of the work are summarized in Section 6.

2. Formulation of the direct problem

We consider an isotropic and linearly elastic FGM panel with a circular hole of radius a subjected to a far-field uniaxial traction σ_0, as shown in Fig. 1(a). We analyze how the radial variation in Young's modulus and Poisson's ratio affects the stress concentration at the hole periphery for plane stress deformations of the panel. We solve the problem by superposing solutions of two problems – biaxial tension and pure shear, as shown in Fig. 1(b) and (c).

We use cylindrical coordinates (r, θ) with origin at the hole center to describe the panel deformations. In the absence of body forces, equilibrium equations are

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0,$$
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2}{r}\sigma_{r\theta} = 0, \qquad (1a,b)$$

where σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are stress components. Hooke's law relating stresses to infinitesimal strains, ε_{rr}, $\varepsilon_{\theta\theta}$, $\varepsilon_{r\theta}$, is

$$\varepsilon_{rr} = \frac{1}{E(r)}[\sigma_{rr}-\nu(r)\sigma_{\theta\theta}],\ \ \varepsilon_{\theta\theta} = \frac{1}{E(r)}[\sigma_{\theta\theta}-\nu(r)\sigma_{rr}],\ \ \varepsilon_{r\theta} = \frac{2[1+\nu(r)]}{E(r)}\sigma_{r\theta}. \qquad (2a\text{-}c)$$

We assume that Young's modulus, $E(r)$, and Poisson's ratio, $\nu(r)$, are given by either

(i) general power-law variations

$$E(r) = E_\infty[1+\beta_1(\frac{r}{a})^n],\ \ \nu(r) = \nu_\infty[1+\beta_2(\frac{r}{a})^n],\ \text{with}\ n<0 \qquad (3a)$$

or

(ii) exponential and power-law variations

$$E(r) = E_\infty exp[\gamma_1(\frac{r}{a})^{-1}],\ \ \nu(r) = \nu_\infty[1+\gamma_2(\frac{r}{a})^{-1}], \qquad (3b)$$

where $E_\infty = \lim_{r\to\infty} E(r)$ and $\nu_\infty = \lim_{r\to\infty}\nu(r)$, β_1, β_2 and γ_1, γ_2 $(-1 < \{\beta_1, \beta_2, \gamma_1, \gamma_2\} < 1)$ are real numbers, and n is a negative integer which helps find an analytical solution of the problem. For a homogeneous material, $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$. The variations of E with r/a for $n = -3, -5$ and $\beta_1 = 0.9, -0.9$ depicted in Fig. 2(a) reveal that $E(r)/E_\infty \to 1$ as $r/a \to 5$. Similarly, the variation of E with r/a for $\gamma_1 = 0.9, -0.9$ depicted in Fig. 2(b) implies that $E(r)/E_\infty \to 1$ as $r/a \to 50$.

The far-field boundary conditions for the biaxial tension and the pure shear problems are:

$$\lim_{r\to\infty}\sigma_{rr}(r) = \frac{\sigma_0}{2} \qquad (4a)$$

$$\lim_{r\to\infty}\sigma_{rr}(r,\theta) = \frac{\sigma_0}{2}\cos 2\theta,\ \ \lim_{r\to\infty}\sigma_{r\theta}(r,\theta) = -\frac{\sigma_0}{2}\sin 2\theta \qquad (4b)$$

and boundary conditions at the hole periphery are

$$\sigma_{rr}(a,\theta) = 0,\ \ \sigma_{r\theta}(a,\theta) = 0. \qquad (4c)$$

When solving the problem for stresses, we employ the following compatibility equation:

$$\frac{\partial^2\varepsilon_{\theta\theta}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varepsilon_{rr}}{\partial\theta^2} + \frac{2}{r}\frac{\partial\varepsilon_{\theta\theta}}{\partial r} - \frac{1}{r}\frac{\partial\varepsilon_{rr}}{\partial r} = \frac{1}{r}\frac{\partial^2\varepsilon_{r\theta}}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\varepsilon_{r\theta}}{\partial\theta} \qquad (5)$$

3. Solution of the direct problem

We analytically solve the problem for the uniform biaxial tension in subsection 3.1, and use the Frobenius series to analyze the problem for pure shear loading in subsection 3.2. By superposing solutions of these two problems, we obtain the solution for the uniaxial tension problem in subsection 3.3.

3.1. Uniform biaxial tension

We note that the problem geometry, the material properties and the

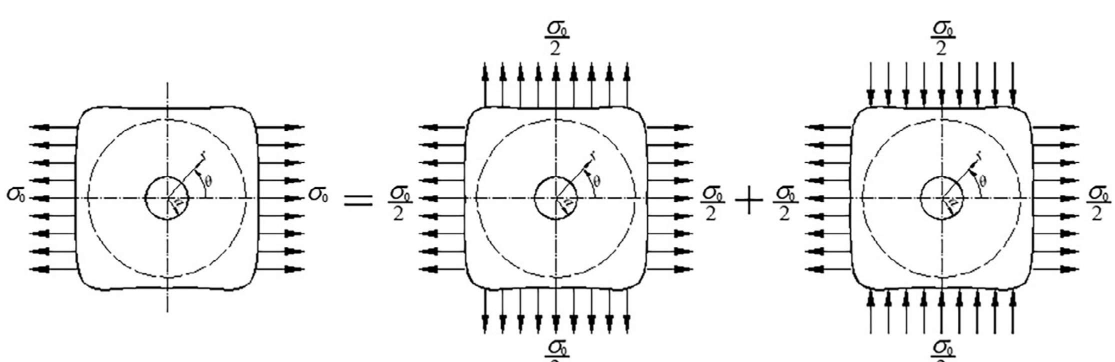

Fig. 1. Schematic sketch of a panel with a circular hole subjected to a uniform far-field tensile traction, and its split into two problems.