Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

Compatibility between fuzzy set operations and level set operations: Applications to fuzzy difference

Hsien-Chung Wu

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan

Received 23 January 2016; received in revised form 1 January 2018; accepted 3 January 2018

Abstract

Three types of binary operation between any two fuzzy sets are proposed in this paper. The concept of 0-level set of fuzzy set plays an important role for defining the binary operations between two fuzzy sets. According to the decomposition theorem of fuzzy sets, the consistency between the membership functions and α-level sets should be rigorously verified when the binary operation between two fuzzy sets is defined by using their α-level sets. The consistency is investigated thoroughly in this paper for the purpose of guaranteeing the existence of such binary operations. Moreover, we also propose the so-called Hausdorff difference between any two fuzzy sets to define the difference of any two fuzzy numbers.

Keywords: Decomposition theorem; Extension principle; Fuzzy numbers; Hausdorff difference

1. Introduction

The difference between any two fuzzy numbers have been studied for a long time. The key issue of improving the fuzzy difference focuses on its existence. Recently, Bede and Stefanini [2] and Gomes and Barros [7] propose the so-called generalized difference between two fuzzy numbers. Since the generalized difference of any two fuzzy numbers that was proposed by Bede and Stefanini [2] cannot be a fuzzy number in general, Gomes and Barros [7] slightly modified the definition of Bede and Stefanini [2] to claim that the generalized difference of any two fuzzy numbers is guaranteed to be a fuzzy number, where the fuzzy number is a fuzzy subset of $\mathbb{R}$ owning the elegant structure such that the α-level sets become the bounded and closed intervals in $\mathbb{R}$. In this paper, we shall generalize the fuzzy difference proposed in Bede and Stefanini [2] and Gomes and Barros [7] by considering the differences of fuzzy subsets of $\mathbb{R}$ instead of fuzzy numbers in $\mathbb{R}$. We shall also study their existence.

Bede and Stefanini [2] and Gomes and Barros [7] used the α-level sets to define the generalized difference of two fuzzy numbers. The generalized differences are fuzzy subsets of $\mathbb{R}$. However the precise expressions of membership functions of generalized differences were not presented by them. According to the decomposition theorem, the membership functions of generalized differences can be obtained by using their α-level sets. Therefore the consistency

E-mail address: hcwu@nknucc.nknu.edu.tw.

https://doi.org/10.1016/j.fss.2018.01.002

between membership functions and α-level sets should be rigorously verified. This verification can guarantee the existence of such generalized differences. However, Bede and Stefanini [2] and Gomes and Barros [7] did not perform this verification, which also says that the existence of generalized differences proposed by them are unknown. In this paper, we shall study the consistency between membership functions and α-level sets by providing some suitable sufficient conditions.

The meaning of 0-level set in fuzzy sets theory is an important issue. Let $\tilde{A}$ be a subset of a universal set U. If the universal set U is endowed with a topology, then the 0-level set of $\tilde{A}$ is defined to be the closure of its support. If the universal set U does not have the topological structure, then the 0-level set is defined to be the whole set U by referring to Negoita and Ralescu [9], and Puri and Ralescu [11]. It is well-known that the family consists of all α-level sets of $\tilde{A}$ is nested. Conversely, we can construct a fuzzy subset of a universal set U based on a family of subsets of U. When the corresponding family is nested, the construction of such fuzzy subset has been studied by Negoita and Ralescu [9], and Puri and Ralescu [11] in which the 0-level set is assumed to be the whole universal set U. The construction of such fuzzy subset has also been studied in Wu [13,14] by considering the 0-level set as the closure of its support. In this paper, we shall propose three types of binary operations considering the issues of 0-level set and non-nested family. These binary operations will be used to define the fuzzy difference that extends the so-called generalized difference proposed by Bede and Stefanini [2] and Gomes and Barros [7].

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of U. We consider the family $\{M_\alpha : 0 < \alpha \leq 1\}$, where $M_\alpha = \tilde{A}_\alpha \circ \tilde{B}_\alpha$ for all $\alpha \in (0, 1]$ and "$\circ$" is a general binary operation defined on the α-level sets $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$. Using the form of decomposition theorem, we can define the general binary operation $\tilde{A} \odot_G \tilde{B}$ with membership function defined by

$$\xi_{\tilde{A} \odot_G \tilde{B}}(x) = \sup_{\alpha \in (0,1]} \alpha \cdot \chi_{M_\alpha}(x),$$

which says that this general binary operation $\tilde{A} \odot_G \tilde{B}$ always exists. The α-level sets of the binary operation $\tilde{A} \odot_G \tilde{B}$ is strongly related with the family $\{M_\alpha : 0 < \alpha \leq 1\}$. If the family $\{M_\alpha : 0 < \alpha \leq 1\}$ is nested, then we can use the results obtained in Wu [14] to present the relationship. If the family $\{M_\alpha : 0 < \alpha \leq 1\}$ is not nested, then the relationship can be obtained from Theorem 3.10 below in the context of this paper. This general type binary operation $\tilde{A} \odot_G \tilde{B}$ that always exists is categorized as the third type of binary operation in this paper. However, the first and second types of binary operations that extend the addition in Puri and Ralescu [11] and the differences in Bede and Stefanini [2] and Gomes and Barros [7] cannot always exist. Their existence needs to assume that the family $\{M_\alpha : 0 < \alpha \leq 1\}$ is nested, which will be realized in the context of this paper.

Based on the form of Hausdorff metric that frequently adopted in set-valued analysis, we can propose the so-called Hausdorff differences that can be used to define the fuzzy difference. We shall claim that the so-called generalized differences proposed by Bede and Stefanini [2] and Gomes and Barros [7] are special cases of Hausdorff differences proposed in this paper. Also, the existence of Hausdorff differences will be studied in this paper.

In Section 2, we propose three types of binary operations that can be used to define the Hausdorff differences. In Section 3, we investigate the existence of these three types of binary operations proposed in Section 2. In Sections 4 and 5, we propose many types of Hausdorff differences and study their existence.

2. Binary operations

Let $\tilde{A}$ be a fuzzy subset of a universal set U with membership function denoted by $\xi_{\tilde{A}}$. In this paper, the universal set U is assumed to be a vector space over $\mathbb{R}$, since we need to consider the difference $a - b$ for any $a, b \in U$. The family of all fuzzy subsets of U is denoted by $\mathcal{F}(U)$. For $\alpha \in (0, 1]$, the α-level set of $\tilde{A}$ is denoted and defined by

$$\tilde{A}_\alpha = \left\{x \in U : \xi_{\tilde{A}}(x) \geq \alpha\right\}. \tag{1}$$

We also define

$$\tilde{A}_{\alpha+} = \left\{x \in U : \xi_{\tilde{A}}(x) > \alpha\right\}.$$

The *support* of a fuzzy set $\tilde{A}$ within a universal set U is the crisp set defined by

$$\tilde{A}_{0+} = \{x \in U : \xi_{\tilde{A}}(x) > 0\}.$$

Please cite this article in press as: H.-C. Wu, Compatibility between fuzzy set operations and level set operations: Applications to fuzzy difference, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.01.002