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Solving fuzzy linear systems using a block representation of generalized inverses I: The Moore–Penrose inverse

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Abstract

In this paper we present a new method for solving an $m \times n$ fuzzy linear system (FLS), $A\tilde{X} = \tilde{Y}$, where the coefficient matrix A is real, using the block representation of generalized inverses. A necessary and sufficient condition for a block matrix to be the Moore–Penrose inverse of the full rank matrix associated to a FLS is given. We obtain a necessary and sufficient condition for the existence of solutions of a FLS, with arbitrary real coefficient matrix. The exact algebraic form, with respect to the Moore–Penrose inverse, of any solution of FLS of this type is established. A general, efficient and universal method for obtaining the exact solutions is introduced. Some numerical examples are presented to illustrate the proposed method.

Keywords: Fuzzy linear systems; Generalized inverses; The Moore–Penrose inverse; Singular matrix

1. Introduction

Systems of linear equations play a major role in various areas such as mathematics, statistics, engineering, etc. In many applications, system's parameters are rather represented by fuzzy numbers than numbers. This fact indicates the importance of developing different methods for solving fuzzy linear systems (FLS), that arose as extensions of a classical linear system $Ax = b$.

There are numerous papers devoted to the investigation of solutions of fuzzy linear systems. FLS are divided into square ($n \times n$) and non-square ($m \times n; m \neq n$). Most works in the literature deal with the square form involving non-singular matrices. The fully fuzzy linear systems, in the form $\tilde{A}\tilde{x} = \tilde{b}$, with all fuzzy entries date back to Buckley and Qu (1991) [9]. Few years after Buckley and Qu, Friedman et al. (1998) [13] introduced a method for solving a square FLS whose coefficient matrix A is a real matrix, while unknown vector and the right-hand side column b are fuzzy number vectors. Influenced by Friedman et al.'s approach, Allahviranlo (2004) [3] and Allahviranlo and Ghanbari (2012) [6] introduced various methods for solving fuzzy linear systems. A method for solving non-square

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FLS was presented by Assady (2005) [7]. Allahviranlo and Kermani (2006) [4] and Abbasbandy et al. (2008) [1] also dealt with this problem and brought to light the importance of the Moore–Penrose inverse of the coefficient matrix in solving non-square FLS.

The idea of a generalized inverse of matrices was considered by Moore (1920) and Penrose (1955) [8,10,17]. The Moore–Penrose inverse of an arbitrary matrix (this generalized inverse is the most popular in the literature and known under different names, among others: the pseudo-inverse, the generalized $\{1, 2, 3, 4\}$-inverse, $\{1, 2, 3, 4\}$-inverse) is defined as a solution of the system of four Penrose's matrix equations, while $\{1, 3\}$-inverse and $\{1, 4\}$-inverse satisfy the systems of two particular Penrose's matrix equations. The overview of theoretical aspects and applications of generalized inverses, especially in solving linear systems, can be found in textbooks [8,10]. Based on the Rohde's method [18], the block representation of generalized inverses was considered in [16].

Friedman et al. (1998) [13] presented the method for solving square fuzzy linear systems and the solution vector called either a strong fuzzy solution or a weak fuzzy solution. This method is questionable by many authors [5,15] because weak fuzzy solutions are not solutions and it was shown in [2] that the sufficient condition (proposed in [13], see also Author's replay [14]) for the existence of unique solutions of square FLS is not a necessary condition. Lodwick and Dubois (2015) [15] highlighted various anomalies in solving fuzzy linear systems arising from lack of understanding of interval linear equations. The focus of our paper is on Friedman et al.'s approach, and we try to clarify this issue in this setting. To our knowledge, in research papers influenced by Friedman et al.'s work, methods for solving a general fuzzy linear system have been given without the exact formulation and clear algorithms for obtaining all solutions. An efficient algorithm for solving square FLS in this setting, with non-singular coefficient matrix A, such that $|A|$ is also non-singular (the entries of $|A|$ are the absolute values of entries of A) has been introduced in [6]. In this case, if a solution exists, it is uniquely determined and identical to Friedman et al.'s strong fuzzy solution. This is a reason why we consider a general fuzzy linear system, with arbitrary real coefficient matrix A in Friedman et al.'s settings, but taking into account results obtained in [6]. Using the block representation of generalized inverses of matrices, our main goals are to:

(1) formulate a necessary and sufficient condition for the existence of solutions;
(2) obtain the exact algebraic form of any solution; and
(3) present an efficient algorithm for determination all solutions of a FLS, whose coefficient matrix is real.

We address multi-criteria decision making problems that can be considered in the settings of Friedman et al.'s approach. In this kind of problems, for each of n elements of some set of criteria, the individual weights (importances) are real-valued and known for all of m possible options (i.e., for each of m options, the linear aggregation function that aggregates n fuzzy values is known), and the aggregated fuzzy value among all criteria is also known for all the options.

The paper is organized as follows. In Section 2, the basic properties, definitions and theorems about fuzzy numbers, fuzzy linear systems and generalized inverses are recalled. In Section 3, a necessary and sufficient condition for a block matrix to be the Moore–Penrose inverse of the full rank matrix associated to a FLS is established. In this section, important theoretical results are proven and their relationships with our main result are highlighted. In Section 4, we present a new method for solving a FLS, whose coefficient matrix is real. The main result is Theorem 8 which gives a necessary and sufficient condition for the existence of solutions and the algebraic characterization of solutions of FLS, with arbitrary real coefficient matrix. The algorithm for solving a FLS of such type based on the Moore–Penrose inverse is given. The proposed method is illustrated by numerical examples in Section 5.

2. Preliminaries

In this section, basic notions about fuzzy sets and fuzzy linear systems are recalled. For more details, we refer the reader to [6,11–13,19]. In Subsection 2.1 we present basic notions about generalized inverses.

Definition 1. A fuzzy set $\tilde{u}$ with a membership function $\tilde{u} : \mathbb{R} \to [0, 1]$ is called a *fuzzy number* if the following conditions are fulfilled:

1. $\tilde{u}$ is upper semi-continuous,

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