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Solving fuzzy linear systems using a block representation of generalized inverses: The group inverse

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Abstract

We present an efficient method for solving a singular, $n \times n$ fuzzy linear system (FLS), $A\tilde{X} = \tilde{Y}$, where the coefficient matrix A is a real matrix, singular or non-singular, using the block structure of the group inverse or any $\{1\}$-inverse. A characterization of the block structure of $\{1\}$-inverses, in particular, the group inverse, but also the Drazin inverse of the matrix associated to a square FLS is given. Based on the presented necessary and sufficient condition for the existence of a solution, the general solution of a square FLS is obtained. Finally, infinitely many solutions of a singular FLS are presented through many interesting examples.

Keywords: Fuzzy linear systems; Generalized inverses; The group inverse; Singular matrix

1. Introduction

Most applications of generalized inverses of matrices have required finding a solution of a linear system of equations [6,7,19]. It is well-known that the general solution of linear systems can be expressed in terms of $\{1\}$-inverses. The most prominent generalized inverse of such type is the Moore–Penrose inverse, originally introduced by Moore (1920) and Penrose (1955). In particular case, the Drazin inverse of a square matrix is identical to its group inverse, and in that special case it is also a $\{1\}$-inverse. Very recently, in the authors paper [15], it has been shown that the Moore–Penrose inverse is the crucial tool for solving fuzzy linear systems (FLS) proposed by Friedman et al. in [11]. In this particular type of FLS, in the form $A\tilde{X} = \tilde{Y}$, the coefficient matrix A is a real matrix, the right-hand side vector is given fuzzy number vector $\tilde{Y}$, and $\tilde{X}$ is unknown fuzzy number vector. The general solution of $m \times n$ FLS of Friedman et al.'s type has been completely characterized by Theorem 8 in [15], and based on that characterization in terms of the Moore–Penrose inverse, an efficient algorithm for obtaining all solutions of FLS has been presented and verified by many examples.

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Friedman et al. (1998) [11] proposed a method for solving a square FLS, whose coefficient matrix is real. The paper [11] has some defects indicated and criticized in [2,13], the first one concerns to the definition of weak fuzzy solutions, and the second one is related to the necessary condition for the existence of solutions of square FLS, what is, in our opinion, finally clarified in [15]. There are numerous of papers inspired by Friedman et al.'s work, most of them deal with non-singular FLS, i.e., FLS with the completely non-singular coefficient matrix (both matrices $A=[a_{ij}]$ and $|A|=[|a_{ij}|]$ are non-singular). Also, there are certain number of papers influenced by Friedman et al.'s approach, among others: Assady (2005) [5], Allahviranlo and Kermani (2006) [3], Abbasbandy et al. (2008) [1], Nikuie (2013) [18], with methods for solving FLS that involve generalized inverses, however those papers do not present a characterization of the general solution, nor algorithms for finding all solutions of FLS. To our knowledge, the first characterization of the general solution of FLS is given in [15], and therein proposed algorithm for obtaining all solutions can be viewed as a proper generalization of the algorithm for obtaining solutions of non-singular FLS proposed in Allahviranlo and Ghanbari (2012) [4].

Although the authors paper [15] presents a practical algorithm for solving $m \times n$ FLS, the natural question arose: Is it possible to characterize the general solution of FLS by any of $\{1\}$-inverses of its coefficient matrix? Therefore, our goal now is to take a closer look on a square FLS and investigate a block structure of $\{1\}$-inverses, in particular, the group inverse, but also the Drazin inverse of its singular associated matrix.

The paper is organized as follows. In Section 2, the definitions of generalized inverses and fuzzy linear systems are recalled, and a method for calculating the group inverse is presented. In Section 3, block structures of $\{1\}$-inverses, the group inverse and the Drazin inverse of the associated matrix to singular FLS are investigated. Some new results regarding to a singular FLS and its associated matrix with index 1 are proven. In Section 4, a necessary and sufficient condition for the existence of a solution of square FLS is obtained, providing a characterization for the general solution in terms of $\{1\}$-inverses. Based on the obtained theoretical results, a method for solving a FLS, whose coefficient matrix is an arbitrary square matrix is proposed. Finally, in Section 4, we give examples of consistent singular FLS with infinitely many solutions and each of those solutions can be expressed in terms of any of $\{1\}$-inverses of the coefficient matrix A. In Section 5, some concluding remarks are given.

2. Preliminaries

We begin by recalling basic notions about generalized inverses (for details, we refer to [6,7,16,19,20]).

2.1. Generalized inverses

Let us denote with $\mathcal{M}^n$ the class of all square $n \times n$ real matrices, I_n denotes the identity matrix of order n, and O denotes the null matrix of order n. Let $\mathcal{M}^n_r$ denotes the subclass of $\mathcal{M}^n$ which contains matrices with rank r, $r \le n$. First, let us recall the system of four Penrose's equations for a matrix F, $F \in \mathcal{M}^n$, given by:

$$FGF = F, \tag{P1}$$

$$GFG = G, \tag{P2}$$

$$(FG)^{\mathrm{T}} = FG, \tag{P3}$$

$$(GF)^{\mathrm{T}} = GF, \tag{P4}$$

where matrix $G \in \mathcal{M}^n$ is unknown [6]. We recall that the index of matrix F, $F \in \mathcal{M}^n$, denoted by $ind(F)$, is the smallest non-negative integer k such that $rank(F^{k+1}) = rank(F^k)$. For each $F \in \mathcal{M}^n$, define $F^0 = I_n$. For a non-singular matrix F it holds $ind(F) = 0$, since $rank(F) = rank(F^0) = rank(I_n) = n$, while $ind(O) = 1$, since $rank(O^2) = rank(O) = 0$ and $rank(O^0) = n$. Two additional equations are defined:

$$FG = GF, \tag{P5}$$

$$F^k GF = F^k. \tag{P6}$$

Definition 1. For any $F \in \mathcal{M}^n$, let $\mathcal{H}\{i, j, \dots h\}$ denotes the set of matrices $G \in \mathcal{M}^n$ which fulfill equations $(Pi), (Pj), \dots, (Ph)$ among the equations (P1) to (P6). A matrix $G \in \mathcal{H}\{i, j, \dots, h\}$ is called an $\{i, j, \dots, h\}$-inverse of F and it will be denoted by $F^{(i,j,\dots,h)}$.