Completeness, metrizability and compactness in spaces of fuzzy-number-valued functions ☆

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Abstract

Fuzzy-number-valued functions, that is, functions defined on a topological space taking values in the space of fuzzy numbers, play a central role in the development of Fuzzy Analysis. In this paper we study completeness, metrizability and compactness of spaces of continuous fuzzy-number-valued functions.
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1. Introduction

Fuzzy Analysis is based on the notion of fuzzy number in the same way as Classical Analysis is based on the concept of real number. From 1986, the so-called *representation theorem* of real fuzzy numbers (see [19]) eased considerably the development of the theory concerning fuzzy-number-valued functions, that is, functions defined on a topological space taking values in $\mathbb{E}^1$, the space of fuzzy numbers. Such functions, as real-valued functions do in the classical setting, play a central role in Fuzzy Analysis. Namely, fuzzy-number-valued functions have become the main tool in several fuzzy contexts, such as fuzzy differential equations ([6]), fuzzy integrals ([36], [40]), fixed point theory ([25,30,39]) and fuzzy optimization ([20], [37], [38]).

In this paper we address three topological aspects of the spaces of continuous fuzzy-number-valued functions when endowed with the most usual topologies. Namely we will study completeness, metrizability and compactness in this context. Only the latter concept, which is clearly related to Ascoli theorem, seems to have received certain attention in fuzzy literature (see, e.g., [32], [14]). The prototype of such result in Classical Analysis was proved by Ascoli in [5] and, independently, by Arzelà, who acknowledged Ascoli's priority in [4]. Nowadays, Arzelà–Ascoli type theorems encompass the study of the (relative) compacity of a family of functions endowed with several topologies and their literature is extense. The applications of these results are numerous in different settings; namely, in the context of

differential equations, in finding extremal curves, in most criteria for the consistency of systems involving inequalities, etc.

In [14, Theorem 4.2], Fang and Xue set up a fuzzy version of Ascoli theorem which characterized compact subsets of the space $C(K, (\mathbb{E}^1, d_\infty))$ of all fuzzy-number-valued continuous functions on a compact metric space K endowed with the topology of the uniform convergence. Unfortunately this version is not correct since, as pointed out in [15], it is based on a wrong characterization ([14, Theorem 2.4]) of the compact subsets of $(\mathbb{E}^1, d_\infty)$. In the last section of this paper we fix [14, Theorem 4.2] by extending the fuzzy Ascoli theorem to a broader framework. The key concepts in our approach are the *bounded subsets* of a topological space and α_f-*spaces*. Thus, previously, in Section 3 we obtain a fuzzy characterization of bounded subsets and α_f-spaces, and point out how such spaces appear in a natural way when considering the completeness of the spaces $C(X, (\mathbb{E}^1, d_\infty))$, X a topological space, equipped with the topology τ_α of uniform convergence on the (bounded) members of a cover α of X. In Section 4, we establish an explicit criterion for $C_{\tau_\alpha}(X, (\mathbb{E}^1, d_\infty))$ to be metrizable. Finally, as mentioned above, in Section 5 we address the fuzzy Ascoli theorem (and also the weak fuzzy Ascoli theorem) for spaces $C(X, (\mathbb{E}^1, d_\infty))$. As a consequence of our results we show that $C(X, (\mathbb{E}^1, d_\infty))$ endowed with the topology of the uniform convergence satisfies the fuzzy Ascoli theorem if and only if X is pseudocompact.

2. Preliminaries and notation

Given a fuzzy subset u on the real numbers $\mathbb{R}$, the λ-level set of u is defined by $[u]^\lambda = \{x \in \mathbb{R} : u(x) \geq \lambda\}$ for $\lambda \in (0, 1]$ and $[u]^0 = \mathrm{cl}_\mathbb{R}\{x \in \mathbb{R} : u(x) > 0\}$ for $\lambda = 0$.

Now, the fuzzy number space $\mathbb{E}^1$ is the set of such u satisfying the following properties:

(1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$.
(2) u is convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$.
(3) $u(x)$ is upper-semicontinuous.
(4) $[u]^0$ is a compact set in $\mathbb{R}$.

Notice that, if $u \in \mathbb{E}^1$, then the λ-level set $[u]^\lambda$ of u is a compact interval for each $\lambda \in [0, 1]$. We denote $[u]^\lambda$ by $[u^-(\lambda), u^+(\lambda)]$. Every real number r can be consider a fuzzy number: indeed, if r can be identified with the fuzzy number $\tilde{r}$ defined by

$$\tilde{r} = \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{if } t \neq r. \end{cases}$$

From now on, we do not distinguish between r and $\tilde{r}$. The following two results are useful in the theory of fuzzy numbers.

Theorem 2.1. [19] *Let $u \in \mathbb{E}^1$ and $[u]^\lambda = [u^-(\lambda), u^+(\lambda)], \lambda \in [0, 1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:*

  (i) *$u^-(\lambda)$ is a bounded left continuous nondecreasing function on $(0, 1]$.*
 (ii) *$u^+(\lambda)$ is a bounded left continuous nonincreasing function on $(0, 1]$.*
(iii) *$u^-(\lambda)$ and $u^+(\lambda)$ are right continuous at $\lambda = 0$.*
(iv) *$u^-(1) \leq u^+(1)$.*

Conversely, if the pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ satisfies the above conditions (i)–(iv), then there exists a unique $u \in \mathbb{E}^1$ such that $[u]^\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0, 1]$.

Theorem 2.2. [19,12] *For $u, v \in \mathbb{E}^1$, define*

$$d_\infty(u, v) = \sup_{\lambda \in [0,1]} \max\left\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\right\}.$$

Then d_∞ is a metric on $\mathbb{E}^1$ called the supremum metric on $\mathbb{E}^1$, and $(\mathbb{E}^1, d_\infty)$ is a complete metric space.